THE ADMINISTRATION OF MATHEMATICS IN SECONDARY SCHOOLS

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ERNST R. BRESLICH

Associate Professor of the Teaching of Mathematics
Department of Education, the University of Chicago

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PREFACE

This is the third of a series of three volumes devoted to discussions of problems related to the teaching of mathematics in secondary schools. The first volume is concerned with problems arising in the choice and use of general teaching procedures and the second deals with specific teaching problems. In the present volume attention is paid to the administrative problems. It aims to assist not only the teachers of mathematics but particularly all those who are preparing to take charge of a department or are in charge of one.

Administrative problems are being classified as related (1) to the direction and supervision of a department and (2) to the curriculum. The first aims to aid the supervisor in becoming increasingly helpful to the teachers in the department, in bringing about departmental unity and co-operation, and in improving teaching. Such supervisory functions as visiting teachers, holding individual and departmental conferences, and training teachers are discussed in detail. Several chapters are devoted to the problems of setting up objectives, formulating a departmental program of testing and measuring, making provision for the individual differences existing among pupils, and developing procedures of remedial teaching.

The curriculum problems are grouped under such topics as methods of selecting materials for teaching purposes; the organization of arithmetic, geometric, and algebraic materials and their distribution; the unification of the various mathematical subjects; the organization of subject matter in pedagogical units; and the relation of the mathematical curriculum to modern trends in education.

The problems were collected by the writer in connection with his duties and functions as supervisor of the Mathematics Department of the University High School. Numerous others were suggested by his students in courses in the curriculum and supervision of mathematics.

The literature has been thoroughly canvassed, and the views of many writers have been presented. Each chapter supplies an exhaustive bibliography of contributions made by various writers.

Wide reading will enable the student to form independently opinions of the best methods of solving the problems arising in the supervision of mathematics and in the selection and organization of the instructional materials.

E. R. Breslich

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CHAPTER I

THE SUPERVISION OF THE DEPARTMENT

Need for supervision.—No more important service may be rendered to a department than that which improves instruction. If instruction is good, the department will succeed even when the equipment is limited, when textbooks are unsatisfactory, and when other unfavorable conditions exist. On the other hand, the most ideal classrooms and the most modern equipment are of little value when the teaching is poor. It is the purpose of this chapter to organize a program of supervision which will be an effective aid to teachers. Improvement of instruction depends to a large extent on guidance, leadership, and on directing and training the teaching staff. A few illustrations will show the need for supervision.

The teaching force is constantly changing. New teachers coming into the department are often poorly prepared for the work they are expected to do. Even when they know the subject which they are to teach, they may have much to learn about how to teach it. They need assistance, encouragement, and inspiration. Without careful direction they will make mistakes that are costly to the pupil and to the school. With the right type of guidance the chances are that they will adjust themselves readily to the ways of the department and of the school in which they are teaching.

Sometimes classes in mathematics are taught by teachers who have made no special preparation for teaching the subject, who are deplorably deficient in content and method, and who need much assistance and advice. No time should be lost to acquaint them with the subject, its objectives, and the best methods of teaching. The supervisor of mathematics will be in the position to render such assistance.

Occasionally teachers are lacking in interest or enthusiasm about the work they are expected to do. They regard teaching as a temporary occupation, as a stepping-stone to some other occupation which they expect to secure later. Until their attitude is changed, their work needs to be supervised most carefully.

It is not always the new and inexperienced teacher who has difficulty and raises serious departmental problems. Sometimes it is the teacher with long experience who has failed to keep up with progress and who cannot become used to modern methods and changes. He needs such assistance as will enable him to adjust himself.

The large high schools require large staffs to teach the classes in mathematics. To avoid confusion it is necessary that the teachers agree as to departmental uniformity, methods, and objectives. Someone who is properly qualified, who has specialized in the subject, who can speak with authority, and who has the confidence of the teachers should be chosen to guide them in matters on which they should come to an agreement.

In most schools general administrative supervision is well formulated and systematized so that it may be easily performed. However, departmental administration, being comparatively recent, is not at all well developed. The functions of the departmental supervisor are not nearly as definitely defined as those of the principal of the school. In small-school systems the superintendent or the principals perform the duties involved in departmental administration. In the larger schools this is impossible since the responsibilities of the principal's office are too numerous. Such a mass of routine and detail is constantly to be disposed of that there is but a small per cent of his time left for supervision. Moreover, the principal of a high school often lacks the preparation required in supervising and directing the work of all the departments of the school. It thus becomes necessary to find persons who are able and trained to assume the responsibility of directing specific departments.

In many schools "department heads" or "chairmen" or "supervisors" are appointed by the administration. In others the teachers of the department elect a member of the staff to assume leadership. Usually the teaching load of the department head or chairman of the staff is reduced to less than that of the other teachers. For he cannot be expected to teach a full schedule of classes and to perform in addition in an efficient manner the numerous duties of the supervisor. On the other hand, it is very desirable that the supervisor be not entirely exempt from teaching. For a person not teaching

any classes will soon be out of touch with the problems which the teachers have to face.

Objections to supervision.—Objections to supervision have come mostly from the teachers. The following are typical: The supervisors are said to be undemocratic and domineering; the criticisms which they offer are resented as unjust and destructive of the individuality of the teacher; they are accused of narrow-mindedness and of overemphasizing the inspectorial functions; their ratings of teachers are said to be unreliable; they are unreasonable in their demands and expect too much of the teachers.

The foregoing criticisms are not really attacking the principle of supervision but are voicing the need of better-trained supervisors. When the supervisor fails to understand the teachers, when the relation of the supervisor to the teachers is merely that of an inspector who gives destructive criticism, the teachers cannot be blamed for objecting to supervision. However, when the supervisor is well trained and helpful to the teachers, objections will cease.

Supervision is not necessarily costly, for the returns from it will more than repay the expenditures. Several studies of supervised and unsupervised teaching have attempted to determine the outcomes of supervision. They tend to show that the results justify the cost. Supervised schools surpass the unsupervised. Supervision encourages better professional preparation of the teachers, and the direction of the staff by an expert is a powerful influence in improving instruction.

Qualifications of the supervisor of mathematics.—To direct the teachers of a department and to help them become more efficient from year to year the supervisor must possess certain qualifications. He should be a person who knows the best techniques of teacher-training. He should be able to tell when teaching is good and when it is poor, to locate exactly the particular teaching performance which is in need of being improved, and to transmit to the teachers the type of criticism which secures results. This requires a thorough knowledge of the subject, of sound educational methods, and an understanding of the difficulties encountered by teachers and by pupils.

The supervisor of mathematics should be familiar with the findings of important investigations relating to the teaching of the sub-

ject, and should be able to demonstrate to the teachers the best ways of performing most effectively the classroom activities which they carry on from day to day. He should be more than an inspector who merely finds fault. His criticisms should be constructive. His attitude should be that of a counselor anxious to help the teachers eliminate the difficulties which arise in the teaching of mathematics. Hence, he should know the common faults and errors of experienced and inexperienced teachers and the most effective ways of helping them to overcome their shortcomings.

Sometimes a teacher's attitude toward supervision is bad because he cannot or will not understand the real purpose of the supervisor's work, or because he dislikes to change methods which he has used successfully in the past. It takes a person with tact and patience to overcome this antagonism. An autocratic attitude is harmful to mutual understanding. When teachers see that they are benefited by supervision, a change in attitude will invariably result. They should be made to feel free to seek the supervisor's advice, and he should be willing to co-operate with them promptly and heartily.

The supervisor should be able to teach skilfully, because he should occasionally give good exhibitions of first-class instruction which the teachers may observe and which should serve as models of teaching. For this reason he should generally do some teaching to keep in close touch with teaching problems.

Ideally, the supervisor should have charge of the whole department from the elementary school on through the secondary school. This tends to bring about an understanding between the teachers of the lower grades and those of the high school. It simplifies the problem of fixing responsibility for the work of each grade where it belongs.

The supervisor should be a person of thorough academic and professional training, able to work out problems arising in connection with the selection and organization of instructional materials and to demonstrate the best methods of presenting them to the pupils. He should know how and where to gather materials for instruction and how to determine the mathematical needs of the school subjects other than mathematics and of the community.

The supervisor is a connecting link between the teachers in the

department and the school administration. Hence, he must possess executive ability and leadership. He should be able to secure and hold the confidence of the administration and to assist the teachers in carrying out educational policies by adapting the general policies of the school to the work of the department. Likewise, good ideas and excellent departmental policies are of little value if the supervisor is unable to enlist the interest, loyalty, and genuine co-operation of the teachers who are to carry out the ideas, and to secure the friendly attitude of a skeptical administration.

The supervisor's advice is likely to be sought in engaging new teachers and in questions of promotion or recommendation for salary increases. He should therefore be fair minded, just, and able to develop standards for judging objectively the work of the teachers.

Classification of the functions of the supervisor.—The specific functions of departmental supervision are numerous. They may be classified broadly as functions relating to administration, teaching, teacher-training, organizing, and research. All types of functions are important.

In administering the work of the department close relationships must be kept between the department and the administrative officers of the school. A friendly attitude of both must be established toward the whole supervisory program.

The teaching and teacher-training functions presuppose that the supervisor is a good teacher and can give examples of teaching that may serve as models to the members of the department.

The organizing and research functions require a person able to enlist the co-operation of the teachers, without which supervision cannot succeed, to organize the instructional materials for teaching purposes, to devise plans for measuring the results of instruction, and to use the findings to make improvements in future teaching.

Administrative functions of the supervisor.—When the supervisor is merely the chairman of the department, his duties are few and easily performed. His responsibilities may be limited to conducting the departmental meetings and making recommendations to the principal as to equipment, repairs, replacements, alterations, and the various needs of teachers and pupils. However, in some schools the supervisor is in reality an assistant to the principal, and in this

capacity he is expected to take over functions that are usually performed by the head of the school. Thus, he is expected to keep the administration fully informed about the quality of work done by the teachers of the department. This calls for conferences with the principal in which departmental matters are brought up that need attention. The supervisor is to transmit the policies of the school to the teachers of the department, and to see to it that they put them into practice. He is in a favorable position to secure the co-operation of the departmental staff on such matters of routine as making reports to the office, securing regularity of attendance of pupils, reducing tardiness, keeping records, and regulating assigned work that will not overburden the pupils.

The supervisor has charge of the departmental meetings, which are to be held with regularity to acquaint the teachers with the best ways of solving the problems that arise in teaching, and to unify the work of the department. For example, if a school undertakes a new teaching procedure, such as supervised study, the details have to be worked out by the various departments, and the supervisor of mathematics has to assume charge in working out the problem for his department.

Since much of the work of a department is done most economically by committees, the supervisor appoints them and directs them in the work they are to perform. He gives them as much assistance as may be reasonably expected and keeps in touch with the progress that is being made.

Attainment of the best results by a department requires an effective teaching technique dealing with such matters as presenting the subject, making assignments, conducting recitations, testing, and doing corrective work. It is the duty of the supervisor to help the teachers become familiar with the best methods of teaching their subject.

Sometimes problems arise out of the relationships between teachers and pupils, and the teachers may have to call upon the supervisor for help. The first year of the teachers new in the department is usually a critical year for them. They are just establishing their reputations among the pupils. They are being tried out as to their ability to deal with difficult situations. Thus, when cases of disci-

pline arise, it is important that the right measures are taken, and the best advice is needed. This usually requires conferences of the supervisor with the individual teachers and in the more serious cases joint conferences in which the supervisor, teacher, and parents take part.

Pupils who enter the school late during a semester present serious problems on account of differences in preparation owing to the differences in the courses of study of schools. It requires careful judgment to place them where they belong, where they may lose least by the change, and where they may work under the most favorable conditions. Sometimes the supervisor and the pupil will need to have a conference to determine where the pupil should be placed.

Slow pupils often present problems which the teacher is unable to solve without advice. If they are in danger of failing, the teacher and supervisor together may be able to work out a plan of corrective measures before it is too late to help them over the critical period. The fact that the case of a failing pupil is studied leads frequently to remedial measures which save him from failing. The problems of the bright pupils should also receive careful consideration of teacher and supervisor.

In dealing with questions relating to selection, promotion, or transfer of teachers, the principal needs to be provided with detailed facts on which to make his decision. The supervisor is in the best position to supply them. He should keep himself fully informed about the teacher's training, personality, growth, attitude, teaching ability and success, and should have other evidence which may aid the principal in making decisions that are fair and that promise to lead to satisfactory solutions of the problems.

Teaching functions of the supervisor of mathematics.—One of the most important functions of supervision is to improve teaching. Because of the numerous duties that fall upon the supervisor there is a tendency to slight the instructional functions. The supervisor should not lose contact with the problems that confront the teacher, and it is a mistake to exempt him from all teaching. He must be able to show to the teachers what constitutes good teaching. Indeed, instruction of high grade done by him should stimulate the

teachers to raise the quality of their own instruction. He may use his classes for teaching demonstrations which the other teachers are to observe.

As a rule, a demonstration lesson should illustrate specific features or steps of the teaching technique. It should provide concrete illustrations of accepted methods and principles of teaching. It should be a model of good teaching. It should illustrate basic principles of teaching and the theory to be mastered by the teachers. It should inspire more than mere imitation. The demonstration may be used to make methods and theories concrete to an extent not possible in a textbook on the teaching of mathematics.

The teachers should be informed in advance as to the aims and purposes of the demonstration lesson and should later have the opportunity for a conference to diagnose, discuss, and interpret the demonstration. Teachers should be encouraged to visit each other for the purpose of seeing demonstrations of specific methods.

Examples of demonstration lessons are not hard to choose. The following are typical illustrations: teaching pupils to solve verbal problems; administering practice in a process after it has been taught; conducting supervised study; maintaining the attention and participation of pupils; making an assignment; teaching silent reading; developing specific habits of study; and motivating a lesson. The supervisor's classroom need not be the only place in which demonstrations are given. Much of this type of work may be done in conferences with individual teachers or in the departmental meetings which all teachers attend. As a rule, the time and place of demonstration lessons should be announced in advance, and all teachers should be invited to be present.

Teacher-training functions of the supervisor of mathematics.—Educational aims and methods are constantly undergoing changes. Hence, experienced and inexperienced teachers need the guidance and direction of an expert who is able to interpret new tendencies and ideas and to show how they are to be used in the everyday work of teaching. This should eliminate the mistakes and waste which arise whenever experimentation is carried on at the expense of the pupil.

New methods should not be forced upon an unwilling teaching staff. Teachers should be led rather than compelled. If they under-

stand the purpose and advantage of a method different from their own, they will respond to suggestions and constructive criticisms. They will be willing to attempt new procedures. If they are observed by the supervisor as they try out new methods in actual practice, further training may be given in individual conferences. Some of the important activities in which training is needed are: using a variety of teaching procedures; constructing instructional tests; diagnosing tests; doing corrective work with pupils and classes; dealing with problem cases; adapting teaching to individual differences; keeping records; and carrying on research studies.

Much of the training of teachers may be done indirectly. They should be stimulated to read and discuss critically magazine articles relating to problems in the teaching of mathematics. Occasionally they should read intensively an important monograph, a thesis, or a study reported in an educational magazine. When questions arise as to the best procedures, the supervisor should direct the teachers to publications in which good procedures are fully discussed. Such reading stimulates professional thinking, growth, and self-improvement. From time to time some teacher should present a review of an important committee report, or of a new book on the teaching of mathematics, or of a new textbook for pupils. Such reports keep the staff informed as to the outstanding contributions found in the current literature, recent tendencies, progressive movements, and the most modern methods.

Professional growth may also be stimulated by providing opportunities to attend teachers' conventions and meetings that are of interest to them. They should be encouraged to become affiliated with one or more of the professional associations in their subject and to take an active part in them. Opportunities for training are offered in correspondence and in summer schools, in which teachers will find courses to aid them in making up deficiencies in their academic or professional preparation.

Not the least important training of teachers relates to the development of a professional attitude. Without it real progress will be retarded, if not checked entirely. When teachers see that they are the first to profit by a change in attitude, it becomes an easy task to secure their co-operation in the undertakings of the department. They should be stimulated to take a growing interest in the

part they are to contribute; to be open-minded toward innovations; to give the policies of the department the attention and backing which they deserve; to perform routine duties willingly; to be liberal with their time for school work; to assume responsibility for the progress of all of their pupils; and to cultivate a spirit of inquiry toward the problems of teaching.

An interesting study aiming to discover the traits which distinguish "superior" from "just satisfactory" teachers was made by the Chicago Principals' Club.[1] The study is divided into six parts: the teacher's classification and training; teaching activity and skill in the classrooms; results of teaching in terms of pupil activity; professional and social characteristics of the teacher; personal characteristics of the teacher; and general characterization of the teacher. The report compiles an interesting list of the twenty-seven most important factors that mark the difference between "superior" and "just satisfactory" teachers.[2]

Preparation of teachers of secondary-school mathematics.—The question of preparation for teaching is raised not only by students who expect to take up teaching as a profession, but also by teachers in service. Training should be threefold: (1) in the field in which the student expects to teach, (2) in fields closely related to the first, (3) in the technique of teaching.

Knowledge of the subject is an all-important prerequisite of every teacher. Much unsuccessful teaching is caused by lack of academic training. The teacher of any grade or course must know more than he teaches. Unless he has an understanding of the relations of the mathematics of that grade to the mathematics of all other grades of the school, he will lack the point of view necessary to make his teaching purposeful and effective. For this reason it is unfortunate to make distinctions between preparation for teaching junior high school mathematics and senior high school mathematics. There are no two types of mathematics in the secondary field. The subject is continuous from the seventh grade on until the junior college level is reached. One of the strong arguments for the junior high school is that it eliminates the gap between elementary mathematics and secondary mathematics. This should not be accomplished, however, at the risk of forming a new gap be-

[1] *Third Yearbook*, pp. 149–86. [2] *Ibid.*, pp. 181, 182, 186.

tween the mathematics of the junior and senior high schools. The senior high school teacher should be fully informed about the work done in mathematics in the lower grades and should plan his courses as a continuation of that work.

Not only should the teacher of secondary-school mathematics be thoroughly familiar with arithmetic, algebra, geometry, and trigonometry, but he should include in his preparation the mathematics usually offered in the junior college. If possible, he should add a good course in calculus. The modern tendency of bringing into the lower courses the simple concepts of the more advanced work demands an understanding and appreciation of the relation of such concepts to the higher work. One who has had a good course in college algebra and theory of equations will be a better teacher of high-school algebra than one who has not taken the courses. The teacher who has taken analytic geometry will have a point of view which is of great value and importance in teaching the graphical work of the junior and senior high school.

A course in plane surveying will acquaint the teacher with some of the most interesting applications of high-school geometry and trigonometry. It also will offer a good review of the two courses.

To give the teacher a background for teaching and an understanding of the development of mathematics a careful study of the history of elementary mathematics is to be recommended. It has been said that the history of mathematics cannot be separated from the history of the race. A knowledge of the history of this subject will thus enable the teacher of mathematics to make his teaching interesting to the pupils and to show them what mathematics means to them personally and to the world as a whole. The foregoing outline constitutes the minimum academic training of those who teach, or expect to teach, secondary-school mathematics.

Preparation for teaching should not be one sided or too highly specialized. Many teachers have to teach more than one subject and should be qualified to give instruction in some courses in a related subject. For the subject of mathematics that would naturally be in the sciences. Furthermore, excellent applications of mathematics are found in science, and a correlation of mathematics and the sciences is very desirable.[3] There is need for more mathe-

[3] See chap. ix.

matics teachers who are trained in physics, chemistry, and mechanics.

It is sometimes said that real teachers are born and not made. This may be true, but even the best will be more efficient if they enter upon actual teaching thoroughly prepared not only academically but also professionally. A knowledge of the technique of teaching and a familiarity with classroom problems enable the teacher to perform his work with confidence. Many are found in the teaching profession who have excellent training in the subject and are failing because they are lacking in professional preparation. It is unfortunate that content and teaching techniques have become completely separated in the preparation of those who prepare to teach. Hence, to the academic training must be added several courses of professional training.

First, an introductory course to the study of education should acquaint the prospective teacher with the development of the American educational system and its divisions from the elementary school to the college, with the origin and development of the curriculum, the methods of determining its content, and the bases for organizing the instructional materials.

A second course should be taken which presents methods of teaching, classroom management, a knowledge of educational theory, and the technique of testing and measuring the results of teaching.

The third course should be educational psychology.

The fourth should be in the field of special methods. It should relate educational theory to the actual teaching activities in mathematics, present best methods of teaching mathematics, the objectives of mathematical training, and the solution of specific teaching problems. It should acquaint the student with the best current textbooks and literature on the teaching of mathematics and the important investigations of teaching problems.

There should be a course in observation and practice teaching. Here the student is introduced to the art of teaching. He gains first-hand knowledge of what pupils are able to understand and to do, and receives training in planning and teaching the units of the course. He learns to make tests, to administer them to pupils, and to evaluate the test materials. Diagnosis and remedial teaching will

acquaint him with the learning process of the pupil. He will develop a wholesome attitude toward the profession, his associates, and his superiors, and will thus be worthy of the high regard of the community and of the school in which he teaches. Finally, every teacher should have a good course in the history of education.

In a number of studies attempts have been made to determine the factors on which teaching success is based. A brief review of the findings is given in the *Third Yearbook* of the Chicago Principals' Club.[4] Most of the investigators attempt to determine by means of correlations the extent to which such factors as personality, scholarship, age, experience, intelligence, health, and numerous others are related to success in teaching.

Visiting the teacher.—It is not safe to depend upon rumors circulated by teachers, pupils, or parents to supply correct information about a teacher's work. A more reliable way is to go into the classroom and to see what is actually going on. Supervision should be based on personal observation, and the fact must not be overlooked that the major aim of visiting the teacher is to find ways to help him, to give encouragement, to stimulate interest in teaching, and to develop growth. Unless this is accomplished, visits of classes are of little use or value.

The supervisor should acquaint the teachers with his plan of visiting and its purpose. They are most likely to do justice to themselves if they understand that the visitor intends to co-operate with them and not to interfere, not merely to make an inspection but to determine and supply their needs. Fear of the supervisor's visits is often the result of the misconception that he is attempting to restrict the teacher's freedom, independence, and growth. Nothing should be left undone to correct this false attitude toward the supervisor's visits. To make supervision effective it must be co-operative and democratic. Some writers feel that this can be accomplished by visiting only upon the request of the teacher. Others are of the opinion that visits should be announced in advance. In both cases the teacher is given time to make careful preparation to show the best work he is capable of doing. On the other hand, an unannounced visit is more likely to reveal a typical classroom situation and to offer the best opportunity for help,

[4] Pp. 143–49.

while an announced visit may defeat the purpose for which it is really made. Perhaps both plans should be used to accomplish best results.

The visit should be as informal as possible. The supervisor, entering the room, finds the class working in the usual way. His entrance is quiet so as to be hardly noticed by the pupils or the teacher. He should take a convenient seat or, if no vacant seat is available, he should stand, like any other interested observer, where he attracts least attention. In no case should class work be interrupted or changed on his account. The teacher should not call attention to the supervisor's presence by introducing him to the class, offering him a seat at the teacher's desk, or asking a pupil to move to make room for him. A nod of recognition is all the supervisor should expect. Later, when the class is busily at work, the visitor may walk about, take a seat in front of the class where he may observe the faces of the pupils, exchange a few words with the teacher, or speak to individual pupils if that seems desirable. However, care should be taken not to disturb the class. When the supervisor leaves, his exit should be as inconspicuous as his entrance, the class work proceeding without interruption.

If, during the visit, the teacher or the class encounter a puzzling, difficult situation, it is usually better to permit them to find the way out of it than to make suggestions or to take charge of the class. The supervisor will thus be able to judge the teacher's ability to overcome difficulties. He may offer his own views to the teacher in conference after class or at some other convenient time. No harm will be done by letting the matter rest, although the conference should be held as soon as possible. It is not advisable that the supervisor demonstrate during the visit methods which he regards more effective than those used by the teacher.

Since the purpose of the visit is to help the teacher, the supervisor must be able to recognize the strong and weak points of the teaching and to remember these well enough to make favorable comments in the first case and to suggest corrective measures in the second. Criticisms should be definite and concrete if they are to be helpful to the teacher. Many supervisors think it advisable to take brief notes during the visit. However, this should be done without distracting the attention of pupils or teacher. Some supervisors pre-

fer not to take notes during the visit but to write their impressions immediately following the visit while they are still vivid in their minds.

The visiting of teachers should begin before too many difficulties have accumulated. The new teachers should be the first to be observed. In general, they will also need to be visited more frequently than the older teachers. However, this is not always the case, and the assistance a teacher needs should determine the number of times the supervisor should visit him. Frequent short visits enable the supervisor to see many different classroom situations and to get a good idea of the type of work done by the teachers. Moreover, they help him to identify the teachers who are in need of more extended visits and assistance.

In these cases the visit should not be too brief. It may not be necessary or even advisable always to stay the whole period, but the teacher should feel that he has been given time to finish what he started to accomplish and that the supervisor remained long enough to see the finish. On the other hand, the supervisor cannot afford to spend time in the classroom when it becomes apparent that there will be no change in the activities of the pupils or in the classroom procedure, and that no further opportunities for constructive criticism are likely to occur. Furthermore, the supervisor should take time to visit the strong teachers as well as the weak members of the staff and to stimulate both to greater effort and accomplishment.

Visitation records.—The visitation record is a device which aims to assist the supervisor in visiting classes. In every visit so many facts are noted that it is impossible to remember them or to keep a written record of all of them. Hence, visitation cards have been devised to simplify matters by classifying the points to be observed under certain major headings. The following illustrations show how this may be done.

1. *The classroom.*—The notes on the room usually refer to light, temperature, ventilation, neatness, equipment, number of pupils, seating arrangement, and blackboard space.

2. *The pupils.*—The visitor takes note of the interest of pupils, the number paying attention, the number participating, their attitude toward the teacher, number and kind of questions asked, evi-

dence of careful and independent thinking, imitation, competition, behavior, habits of study, use of good English, desire to learn, consciousness of the goal of the lesson, assumption of responsibility, understanding, character of written work, condition of notebooks, and care of equipment used in the work.

3. *The teacher.*—The personality record of the teacher is concerned with such data as health, vitality, appearance, dress, voice, poise, leadership, academic and professional preparation, manners, tact, leniency, exactness, sense of humor, open-mindedness, sincerity, loyalty, sympathy, enthusiasm, and friendliness.

4. *Classroom management.*—This refers to opening and closing the class period, collecting and distributing materials, showing evidence of careful planning, using class time economically, keeping records, correcting the conduct of the class, insisting on accurate language, meeting individual differences, and keeping pupils busy and interested.

5. *Teaching technique.*—Observation should be concerned with the teacher's success in establishing relations between the old and new work; giving summaries, reviews, practice, and drill; asking the right kind of questions; using models; making clear explanations; teaching pupils to study; providing for individual differences; conducting a recitation; lecturing to pupils; giving all pupils opportunity for work; and testing understanding.

6. *Teaching skill.*—The skilful teacher shows his ability to incite learning, to secure interest, to motivate the work, to ask thought-provoking questions, to stimulate initiative, to remove the pupils' difficulties, to look after individual needs, to place emphasis where it belongs, to use his time to the best advantage, and to avoid unnecessary talking.

7. *Subject matter.*—Facts like the following may be recorded: the extent to which the text was followed or to which the material was made to fit the needs and abilities of the pupils; the thoroughness of organization of materials; and the pupil's introduction to the new work taken up by the teacher.

A visitation record blank is easily constructed by organizing carefully the foregoing points and others which may seem worth while to the supervisor and which he may readily observe. It is not intended that in every visit all of the points are to be recorded.

Nor is it necessary to take the card into the classroom for checking, since the marking of the card might disturb the teacher. The real value of it is that it helps the supervisor to make his visits purposeful, intelligent, and worth while. If he has a record card that satisfies his needs and if he is thoroughly familiar with it, he will avoid snap judgments and will have no difficulty in noting and watching for points that should be discussed with the teacher to bring about improvement of instruction. The card may be filled out some time after the visit, and only the most important points need to be recorded. A visitation record card for mathematics has been devised by Breckenridge.[5] Johnson suggests the following helpful outline:

I. Classroom Management
II. Selection and Arrangement of Subject Matter
III. The Recitation
  A. Aims
  B. Division of the Recitation Period
  C. Teaching Devices[6]

Conferences on visits.—The supervisor's visits will be increased in value to the teachers if they are followed by conferences. The length of a conference depends largely on the number of points to be discussed. If there are no important criticisms, a few comments during or after the class period may be all that is needed. The teacher is entitled to some comments which reflect the supervisor's impression of what he observed during the visit. Requests for conferences are sometimes made by the teachers themselves when they feel that they are in need of advice. Indeed, they should be made to feel free to make such requests.

When teachers have serious difficulties, lengthy conferences may be necessary. It is wise not to let them come immediately after the visits. When a teacher is not accustomed to visits or was unsuccessful, he has been under a severe mental strain during the visit and is in no mood to profit by criticisms. For the time being the supervisor should be satisfied with a few friendly comments and arrange for a conference at a later time. However, he should not postpone

[5] William E. Breckenridge, "Judging a Teacher of Mathematics," *Mathematics Teacher*, VIII (June, 1916), 169–72.

[6] F. W. Johnson, "The Supervision of Instruction," *School Review*, XXX, (December, 1922), 742–54.

it too long, since there is danger of forgetting the specific situations or activities on which criticisms are to be given.

The supervisor should go into the conference carefully prepared. He should have before him notes on the happenings that occurred during the visit on which he intends to give constructive criticisms, and which are to be used as bases for improving instruction. If there are too many things that need to be discussed, it is best to select a few of the most important. It has been found helpful to give the teacher a copy of the notes to enable him to go over the various points later.

Criticisms should be favorable as well as adverse. They should be specific and concrete. Vague criticisms should be avoided. With the objective evidence in hand the supervisor need not hesitate to be frank. At the same time a dogmatic attitude is to be avoided. The aim is to secure co-operation and improvement, and to lead the teacher to the understanding and solution of his problems. He should feel that the conference has been of genuine help. The relationships between teacher and supervisor should remain cordial even if unfavorable criticisms have to be given. The teacher should feel encouraged by the conference.

Often a conference should be followed by a second visit to determine to what extent improvement has been made. Other conferences may be necessary if the teacher has failed to profit by the suggestions given in the first.

Rating of teachers.—Improvement of instruction should be the major purpose of teacher-rating. If this is made clear to the teachers and if the supervisor is consistently making use of the findings of his rating scheme in the conferences with the teachers to aid them to overcome their specific difficulties, their attitude toward this phase of supervision will constantly improve. To be sure, rating may be of assistance in questions relating to promotion, salary, retention, and dismissal, but it should be only one of several facts to be used in reaching decisions in such matters. If supervisors abuse the rating of teachers by employing it entirely for purposes other than improvement of instruction, the teachers' attitude will be that of dislike and distrust, and its most important values will be lost.

No matter what rating scheme is used, misunderstandings on the

part of the teachers as to its uses and purposes will prove fatal to its success. Teachers should go over the scheme until it is understood. They will thus see that it is not only helpful to them in bringing about improvement in their work, but that as an instrument of judging the quality of teaching it is to be preferred to a subjective opinion of the supervisor. Careful rating replaces vague opinions by an objective appraisal of performance and is safe and fair. Moreover, the teachers should be shown their ratings. When they know the points on which they are being rated, they learn what is expected of them and are therefore able to improve. For this reason rating should not be deferred too long and should be repeated during the year. If the teaching is unsatisfactory, if weaknesses have developed, the sooner they are brought to the teacher's attention the better. The result will be co-operation and improvement.

Rating scales for teachers have been constructed to assist the supervisor in evaluating teaching performance. They usually contain lists of characteristics that are considered essential to teaching success. The most serious difficulty in the use of such scales arises from the fact that too many qualities are listed. They cease to be practical and useful because of their complexity. It is extremely difficult to give accurate ratings on too many items. Hence, the results of the scale are questionable. Nevertheless, a study of several scales is advisable because it strengthens the supervisor's ability to rate teachers. An exhaustive list of teacher traits is given by Charters and Waples[7] and another has been made up by the Chicago Principals' Club.[8] Other lists are found in the bibliography at the end of this chapter.

In the use of such lists it is advantageous to collect the items under a few major traits such as appearance, health, personality, mental ability, co-operation, enthusiasm, leadership, citizenship, attitude, progressiveness, resourcefulness, scholarship, discipline, self-control, academic training, professional skills and interests. The teacher's attention is then called to traits in which he is lacking

[7] W. W. Charters and Douglas Waples, *The Commonwealth Teacher Training Study* (Chicago: University of Chicago Press, 1929).

[8] "A Study of the Factors That Characterize Superior Teaching," *op. cit.*, pp. 141–86.

and he is shown the characteristics which enter into them. Such analysis brings out the specific phases that need to be developed to secure improvement.

A study of the traits essential to teaching success trains the teacher to rate himself. Self-rating should be encouraged as a means of determining personal deficiencies and of developing professional growth. There is no reason why the teacher should not learn to use lists of traits for self-analysis and for rating his own work. For example, if he wishes to rate himself in regard to "Co-operation" he may ask: Have I been loyal to the school? Have I been doing more than is required? Have I volunteered assistance? Have I supported the policies of the department? Have I contributed to the discussions with constructive criticisms? What may I do to show my willingness to co-operate?

A rating on "Success in Planning Work" may be obtained from questions like the following: Was the purpose of the lesson clear? Was the work motivated? Was the subject matter properly selected and organized? Was sufficient emphasis placed on the essentials?

The teacher may rate himself as to his "Skill in Presenting Units of Instruction" by such questions as: Was I able to gain and hold the attention and interest of all pupils? Did I attain definite results? Was I economical with the time at my disposal? Were individual differences provided for?

The following questions will aid in rating "Professional Attitude": Are my professional standards high? Am I open to suggestions and criticisms? Are my relationships with teachers and officers of the school cordial? Am I co-operating with the administration and the parents of the pupils? After rating himself the teacher may compare his own rating with that of the supervisor. A conference with the supervisor in regard to the abilities on which the two ratings do not agree will be certain to aid in improvement.

Rating tests.—Attempts have been made to develop tests for measuring factors of importance for success in teaching. The tests may be used in selecting candidates for teaching positions and may serve as bases for discussion of teaching procedures in departmental meetings. They may also be used by the teachers to rate themselves. Published rating scales of the type described on page 17 are helpful but not essential. It is an easy matter for a supervisor to

collect a number of typical teaching and classroom problems and to use them as items for a rating test. The following questions illustrate the type of material that is suitable:

1. What should you do if two pupils carry on a continuous conversation as you are teaching a lesson?

Answer:..

2. What should you do if you summon a pupil for a conference and he fails to keep the appointment?

Answer:..

3. The teacher should spend most of his time during supervised study helping the slowest pupils in the class. *True:*............; *False:*............

4. If the pupil fails in an examination the teacher should recommend to the parents that the pupil be tutored until he passes the examination. *True:*............; *False:*............

Like the rating scales, the teacher may use the rating tests to test himself. An interesting attempt along this line is reported by Franzén.[9] He constructed a list of forty-four teaching activities and forty-five pupil activities which could be used as a self-checking device by the teacher and as a visitation record by the supervisor.

Organizing functions of the supervisor.—One of the most pressing organizing functions of the supervisor is the construction of the course of study. Many investigations have given evidence of the poor results obtained in the study of the traditional courses in mathematics. They call for a thoroughgoing reorganization. It is the duty of the supervisor to formulate a clear statement of objectives of the courses which take into consideration the needs and interests of individual pupils, of various types of schools, and of the community. This requires a large amount of study and reading. Another problem is to select and organize the instructional materials for teaching purposes.[10] The general plans are developed by the supervisor, but the amount of detailed work that needs to be done is so large that the teachers should share the burden. It calls for the organization of committees to work under the direction of the supervisor.

After the course of study has been made there is still much to

[9] Carl G. F. Franzén "Improvement Sheet for Algebra," *School Science and Mathematics*, XXXII (December, 1932), 939–43.

[10] See chaps. vi, vii, viii.

be done. Assignments must be differentiated to make allowance for the individual differences which exist within every normal group of pupils.[11] Time schedules have to be organized to regulate the progress of individuals and classes. Provision must be made for the interests of special groups, such as the commercial, industrial, college preparatory, and the group of pupils expecting to leave school at the end of the year.

A variety of schemes is available for adjusting the course to individual differences. Among the best known are individual instruction, supervised study, projects, contracts, maximum and minimum assignments, and grouping. No matter which plan is used, it needs to be carefully organized. Thus, if the plan of grouping is chosen, questions arise as to the best bases for grouping. Is an intelligence test sufficient to group pupils for algebra? Should it be supplemented by an arithmetic test or an algebra prognostic test? What use may be made of a reading test, or of the grades of the pupils given by the teachers of the preceding courses? Should combinations of several devices be employed? If so, which should be selected? After the plan for grouping has been organized it is still necessary to provide for transfers of pupils who have been placed in sections in which they do not really belong.

A testing program needs to be organized to enable the teachers to measure the results of teaching, to make diagnoses, and to do corrective work.[12]

The supervisor should develop keys to be used in the selection of textbooks.[13] After the books have been chosen they are to be adapted to the course of study. This involves preparation of syllabi, outlines, and schedules to help the teachers in the use of the books.

The value of the departmental meetings will be greatly increased if careful plans are made and helpful programs organized. Unless they contain matters of interest to all, the teachers will soon regard the meetings as a burden and a loss of time.

The foregoing are but a few illustrations of the many organizing functions the supervisor has to perform. To be sure, the teachers

[11] See chap. iii.

[12] See chap. ii.

[13] See chap. iv.

must help carry the load, but on the whole the planning and organizing will have to be done by the supervisor himself.

The department meeting.—Department meetings offer the most convenient and effective means to stimulate the desire for improvement and professional growth. Problems of concern to the entire department should be discussed and an agreement should be reached as to the best methods of solution. Thus, departmental unity will be maintained and the desire for improvement and professional growth stimulated.

The meetings should be held at regular intervals, at specified time and place. Experience tends to show that for a high school of five hundred pupils, or more, there are enough matters of importance to justify one meeting each week to dispose of the problems of interest to the mathematics department. Some of the meetings should be for elementary and secondary teachers of mathematics. This will establish an understanding between the two divisions and bring them into close relationships to each other.

All meetings should start and close as promptly as the class periods of the school. It is unfair to a busy teacher to be kept waiting for a meeting to begin or to be detained at the end of a meeting just because others cannot or will not appear on time. The length of a class period, i.e., forty-five or fifty minutes, is sufficient for disposing of matters that need to be taken up. It is unwise to extend the time or to dismiss only those who cannot remain after a busy meeting. Important matters deserve the attendance of all teachers and should not be settled by a few. Time may be saved by disposing briefly of purely individual difficulties and problems. Special conferences may then be arranged for further discussions. Thus, almost the entire meeting will be devoted to problems of interest to all teachers, and all of them will feel that every meeting is of direct value to them. Time may also be saved by use of mimeographed copies of matters relating to departmental routine, equipment, committee reports, school policies, administrative requests, and general announcements. All that is necessary for the supervisor is to supplement the mimeographed statements by brief comments and to answer questions that may be raised by the teachers.

It is as important that the full time of the department meeting

be used as it is not to hold teachers overtime. If there is a surplus of time the supervisor should be prepared to make good use of it. For this purpose he should keep on hand useful information gathered during class visits, or he may discuss departmental policies and plans for the future.

Department meetings may be held in a classroom or in a room especially designed for conferences. The advantage of the first is that it is equipped with all materials and instruments that may be needed. The disadvantage is that the meetings are likely to become formal and tiring. A conference room offers a pleasant change from classroom work and gives the meeting a less formal aspect. However, it will not be as effective as the classroom unless it is completely equipped with blackboard, compasses, ruler, crayon, textbooks, and other necessary materials.

It is not always easy to find a time for the meeting that is satisfactory to all of the teachers. In small high schools the supervisor is often permitted to work out the programs of the teachers in a department in a way which leaves them free during one of the class periods of the school day. When this cannot be done the next best alternative is to meet before the beginning or at the end of the school day. Evening or Saturday-morning meetings are not recommended. They deprive the teachers of time which should be used for study, marking papers, planning school work, and community or family activities. The determination of the time of the meetings may be left to the teachers. The question should be decided in the first meeting of the year.

The supervisor may add to the value of the department meetings by providing for discussions of a number of major teaching problems. A program should be planned for the whole year and copies should be distributed to the teachers at the first meeting. If the problems are selected from lists handed in by the teachers, the effectiveness of the plan may be greatly increased. The most important problems are chosen and one is assigned to each meeting. If references to textbooks and magazine articles are supplied, the teachers will be able to study the problem in advance. They will come to the meeting prepared to offer their opinions and experiences. The discussion should be led by one of the teachers, and all should be encouraged to participate.

Another way of selecting the problems is to take them from textbooks on the teaching of mathematics. By this method a systematic study of such books may be made, and teachers may thus be stimulated to continue to read the literature in their special field. The following problems are suitable for discussion in department meetings: (1) how to use various methods of teaching, e.g., the laboratory method, the genetic method, the experimental method, or the project method; (2) how to choose the objectives of mathematical instruction, select the most suitable material, and organize it for teaching purposes; (3) how to teach certain specific processes, facts, and principles, such as factoring, signed numbers, graphs, geometric constructions, and solution of original exercises; (4) how to measure the results of teaching by unit tests and standardized tests, how to make diagnoses of test results, and how to do remedial work with the pupils; (5) how to reduce failures in the department. A list of problems is given in the writer's *Problems in Teaching Secondary School Mathematics*, chapter i.

Occasionally a department meeting may be devoted to the comparison and evaluation of various procedures aiming to solve a specific teaching problem. The development of judgment of the relative effectiveness of several procedures is an important phase of teacher-training. The material may be taken from a classroom procedure test.[14] The test is first administered to the teachers to insure careful examination of the suggested procedures. The strong and weak points are then discussed and a departmental agreement on the most and least efficient procedures is reached. The following is taken from the procedure test:

SUPERVISING STUDY

A class in algebra has been taught a new principle, and the teacher has assigned a number of exercises to be worked out during the class period for the purpose of giving the pupils experiences in applying the principle. Assume that after working one or two minutes one of the pupils raises his hand and requests assistance. The following procedures may now be used:

[14] E. R. Breslich and Charles A. Stone, *Class Room Procedure Test for Teachers of Mathematics* (Chicago: University of Chicago Press, 1928).

1. The pupil's appeal is ignored and he is admonished to try harder.
2. The teacher goes to the pupil and works with him until he says that he understands the work.
3. One of the bright pupils of the class is asked to help him overcome his difficulty.
4. The teacher refers the pupil to a place in the textbook where a similar problem is worked out, and asks him to study it.
5. The teacher examines the pupil's written work, gives him a suggestion, and tells him to try again.

Decide which of the five procedures is the most efficient way to meet the situation and which is the least efficient.

Some of the meetings may be used for demonstration lessons. Following the discussion of the advantages and disadvantages of various ways of teaching a topic, the supervisor demonstrates the particular technique which the department should adopt in the future. Examples of lessons to be demonstrated are the teaching of graphical solution of equations, solution of verbal problems, a type form in factoring, subtraction of signed numbers, geometric constructions, the meaning of the trigonometric ratios, the use of mathematical instruments and the logarithmic table.

Many problems arise on which departmental unity of action is desired. Teachers need to be shown what to do in certain well-defined cases of discipline and dishonesty, how to assist pupils that are failing, how to mark the written work of pupils, and how to help pupils remove specific deficiencies. Such problems should be presented for discussion to aid teachers in avoiding serious mistakes.

The department meeting should be used to stimulate professional growth. Hence, places should be provided on the program for reviews of the most recent textbooks, important research studies, interesting articles published in the mathematical journals, and the latest books on the teaching of mathematics. Occasionally, when a difficulty is brought up in a meeting, the teachers should be referred to books or articles in which the problem has been discussed. They may thereby be stimulated to read the mathematical literature.

The first department meeting.—In most schools a general faculty meeting is held one or two days before school opens. There is much

to be said and done if a good start is to be made in all classes. The practice should be adopted by all departments of the school. The teachers need information about the classes assigned to them, the equipment to be supplied to the pupils, the textbooks to be used, and the teaching technique to be employed. Schedules for the first units should be presented. The teachers will raise questions in regard to their work which should be answered by the supervisor. Schemes for grouping pupils, the testing program, and remedial teaching should be explained. This is the time to appoint committees and to assign special services to be rendered by the teachers to the department and the school. Thus, there is need for committees on textbooks, curriculum, equipment, and current mathematical literature. Advisers must be assigned for the mathematics club, for pupils preparing for special examinations, and for pupils working on supplementary or voluntary projects. Perhaps some teacher should be selected to supervise an after-school opportunity class for pupils who wish to make up back work or remove specific deficiencies. This load of departmental responsibilities should be evenly distributed among the teachers.

Furthermore, the administration of the school has the right to expect services from the teachers. Standing committees are to be formed. The literary clubs and school publications must be guided. Study halls have to be taken care of. Every teacher must contribute his share to the non-departmental phases of school work. It should be understood that in making such assignments the special interests and abilities of the teachers should be taken into consideration.

Occasionally a teacher prefers to spend his time on some research problem. He should be required to prepare a careful statement of the problem, method of treatment, sources and data, facilities needed, and a tentative schedule of the work from the beginning to its completion. If the statement is satisfactory and the problem deserves it, the supervisor should render all possible assistance in securing special classes, equipment, clerical service, and materials that may reasonably be supplied.

A careful record extending over a period of years should be kept of all services rendered by the teachers in order that this type of

work be properly distributed. The accompanying form has been found convenient for the purpose.

Name of Teacher	No. of Classes	Departmental Services	Special Services to the School	Research Problem

Copies of the plan of the meetings to be held during the semester should be distributed among the teachers at the first meeting, or as soon as possible. The schedule should then be observed rigidly. A change in the time of the meetings or an omission is likely to interfere with other plans of the teachers. This usually results in irregularity of attendance and dissatisfaction.

BIBLIOGRAPHY

TEXTBOOKS ON SUPERVISION

Alberty, H. B., and Thayer, V. T. *Supervision in the Secondary School.* New York: D. C. Heath & Co., 1931.

Barr, A. S., and Burton, W. H. *The Supervision of Instruction.* New York: D. Appleton & Co., 1926.

Burton, W. H., and Barr, A. S. *Supervision and the Improvement of Teaching.* New York: D. Appleton & Co., 1922.

Carpenter, W. W., and Rufi, John. *The Teacher and Secondary School Administration from the Point of View of the Classroom Teacher.* Boston: Ginn & Co., 1931.

Chancellor, W. E. *Our Schools: Their Administration and Supervision.* New York: D. C. Heath & Co., 1915.

Collings, Ellsworth. *School Supervision in Theory and Practice.* New York: Thomas Y. Crowell Co., 1927.

Cubberley, E. P., *The Principal and His School.* Boston: Houghton Mifflin Co., 1923.

Jessup, W. A., and Coffman, L. D. *The Supervision of Arithmetic.* New York: Macmillan Co., 1917.

Johnson, F. W. *Administration and Supervision of the High School.* New York: Ginn & Co., 1925.

Kyte, George C. *Problems in School Supervision.* Boston: Houghton Mifflin Co., 1931.

Maxwell, C. R. *The Observation of Teaching.* Boston: Houghton Mifflin Co., 1917.

Nutt, H. W. *Current Problems in the Supervision of Instruction.* Richmond: Johnson Publishing Co., 1928.

———. *The Supervision of Instruction.* Boston: Houghton Mifflin Co., 1920.

Reavis, W. C., Pierce, Paul R., and Stullken, Edward H. *The Elementary School.* Chicago: University of Chicago Press, 1931.

Reeves, Charles E. *Standards for High School Teaching: Problems in Methods and Technique.* New York: D. Appleton & Co., 1932.

Smith, David E., and Reeve, William David. *Teaching of Junior High School Mathematics,* pp. 293–96. Boston: Ginn & Co., 1927.

Strayer, G. D., Engelhardt, N. L., and Others. *Problems in Educational Administration.* New York: Bureau of Publications, Teachers College, Columbia University, 1925.

Wagner, C. A. *Common Sense in School Supervision.* Milwaukee, Wis.: Bruce Publishing Co., 1922.

Waples, Douglas. *Problems in Class Room Method.* New York: Macmillan Co., 1927.

Uhl, Willis L. *The Supervision of Secondary School Subjects.* New York: D. Appleton & Co., 1929.

ARTICLES DEALING WITH GENERAL SUPERVISION

Anderson, W. N. "Some Elements of School Supervision," *American School Board Journal,* LXVII (November–December, 1923), 39–40, 48–49.

Armentrouth, W. D. "Supervision and Educational Aims," *Journal of Educational Method,* II (March, 1923), 272–81.

Ashbaugh, E. J. "Some Essentials in School Supervision," *Journal of Educational Research,* VI (September, 1922), 116–25.

Bamberger, F. E. "Supervision: A Forward Look," *School and Society,* XXIV (December 18, 1926), 747–52.

Barr, A. S. "Objective Supervision," *Journal of Educational Research,* XIX (March, 1929), 201–2.

Belser, D. "Problems in Supervision," *National Education Association Proceedings* (1929), pp. 807–8.

Bobbitt, Franklin. "Mistakes Often Made by Principals," *Elementary School Journal,* XX (January and February, 1920), 337–46, 419–34.

Breed, F. S. "Remedial Supervision Based on a Diagnostic Survey of Instruction," *National Conference of Supervisors and Directors: Second Yearbook*, pp. 65–76.

Brim, O. G. "Changing and Conflicting Conceptions in Supervision," *Journal of Educational Method*, X (December, 1930), 131–40.

Brink, William G. *Direction and Coordination of Supervision*, "Northwestern University Contributions to Education," No. 3. Bloomington, Ill.: Public School Publishing Co., 1930.

Brueckner, L. J., and Cutright, Prudence, "A Technique for Measuring the Efficiency of Supervision," *Journal of Educational Research*, XVI (December, 1927), 323–31.

Campbell, B. "Establishing Supervisory Technics," *Department of Elementary School Principals' Bulletin*, X (April, 1931), 540–43.

Chicago Principals' Club Report. *Supervision*, Bull. I (1932).

———. *Ibid.*, II (1924), 1–30.

Cody, Frank L. "Why Is a Supervisor?" *American School Board Journal*, LX (March, 1920), 54–117.

Courtis, S. A. "Measuring the Effects of Supervision," *School and Society*, X (July, 1919), 61–70.

Dondineau, A. "Organization of Supervision in Large City Systems," *Journal of Educational Method*, IX (May, 1930), 467–72.

Dunn, Fannie W. "The Distinction between Administration and Supervision," *Educational Administration and Supervision*, VI (March, 1920), 159.

Editorial. "Value of Supervision," *Elementary School Journal*, XXVII (April, 1927), 564–65.

Gambrill, B. L. "Critical Review of Researches in Supervision," *Educational Administration and Supervision*, XV (April, 1929), 279–89.

Gerling, H. J. "A Superintendent's Viewpoint," *Department of Elementary School Principals' Bulletin*, X (April, 1931), 175–77.

Gray, William S. "The Work of the Elementary School Principal," *Elementary School Journal*, XIX (September, 1918), 24–35.

Hart, Melvin C. "Supervision from the Standpoint of the Supervised," *School Review*, XXXVII (September, 1929), 537–41.

Hosic, J. F. "Balanced Program in Supervision," *Educational Method*, VIII (May, 1929), 445–49.

———. "Fourth Yearbook: Department of Supervisors and Directors of Instruction," *School and Society*, XXXII (December 6, 1930), 766–67.

Hughes, J. M. "Study in High School Supervision," *School Review*, XXXIV (February, 1926), 112–22.

Johnson, F. W. "The Supervision of Instruction," *ibid.*, XXX (December, 1922), 742–54.

Judd, C. H. "The Principal as a Supervisor of Classroom Teaching," *National Education Association* (1926), pp. 825–31.

Knight, F. B. "Possibilities of Objective Techniques in Supervision," *Journal of Educational Research*, XVI (June, 1927), 1–15.

Koch, Harlan C. "Some Aspects of the Department Headship in Secondary Schools, "*School Review*, XXXVIII (April, 1930), 263–75.

Lewis, E. E. "Scientific School Supervision," *American School Board Journal*, LXVI (February, 1923), 43–44.

Lindquist, R. D. "Evaluation of Supervision," *Educational Administration and Supervision*, XV (April, 1929), 301–10.

Maxwell, C. R. "Effective Supervision," *School and Society*, XI (February 21, 1920), 214–16.

Montgomery, Rhoda. "The Principal's Supervision of the Special Subjects," *Sixth Yearbook: Department of Elementary School Principals* (National Education Association, 1927), pp. 168–77.

National Conference of Supervisors and Directors of Instruction. *Scientific Method in Supervision: Second Yearbook.* New York: Bureau of Publications, Columbia University, 1929.

Oertel, Ernest E. "Creative Supervision vs. Inspection," *American School Board Journal*, LXXVIII (June, 1929), 40.

Puckett, R. C. "Making Supervision Objective," *School Review*, XXXVI (March, 1928), 209–12.

Reeves, C. E. "Principal as Supervisor," *American School Board Journal*, LXXVIII (February, 1929), 51–52.

Sloyer, M. W. "Subject Supervision," *Education*, XLVIII (April, 1928), 465–79.

Stevens, L. B. "A Supervisor of Arithmetic Sketches a Typical Day's Work," *Journal of Educational Method*, VIII (January, 1929), 220–25.

Thompson, H. G. "Principals' Conferences as an Aid to High School Supervision," *High School Quarterly*, XVIII (April, 1930), 144–47.

Twitchell, D. F. "Objective Measure in Supervision," *Journal of Educational Research*, XIX (February, 1929), 128–34.

Wagner, C. A. "Arguments for and against the Supervision of Instruction," *National Education Association* (1922), pp. 1438–39.

———. "Some Types of Misconceived Supervision of Instruction," *American School Board Journal*, LXVI (May, 1923), 37–38.

———. "Why So Little Supervision?" *ibid.*, September, 1923, p. 49.

Waples, Douglas. "Teaching Teachers To 'Motivate,'" *Educational Administration and Supervision*, VII (November, 1921), 439–46.

Wegner, H. C. "Practical Outline Governing Principles of Supervision," *American School Board Journal*, LXXX (March, 1930), 53–54.

Woody, C., and Others. "Evaluation of Supervision," *Journal of Educational Method*, X (April, 1931), 398–403.

SECONDARY-SCHOOL SUPERVISION

Allen, Charles F. "A Three Year Program of Supervision," *High School Teacher*, I (September, 1925), 244–45.

Aseltine, John. "The Duties of a Department Head in a Large City High School," *School Review*, XXXIX (April, 1931), 272–79.

Ashbaugh, E. J. "Supervision of High School Instruction," *High School Teacher*, I (June, 1925), 180–81.

Backus, Bertie. "The Supervision of Instruction in the High School," *Educational Administration and Supervision*, II (February, 1925), 112–17.

Bobbitt, F. "Supervisory Leadership on the Part of the High School Principal," *School Review*, XXVII (December, 1919), 733–47.

Bossing, N. L. "Development of the Supervisory Idea in Secondary Education," *High School Journal*, VIII (November, 1930), 31–35.

Briggs, Thomas H. "Responsibility of Supervision," *High School Teacher*, V (October, 1929), 270–72.

Carr, L. C. "Work of a Woman Assistant Principal in a Six-Year High School," *School Review*, XXXVIII (November, 1930), 700–706.

Clement, J. A. "Broadening Conception of Supervisory Leadership in Grades 7 to 12," *High School Quarterly*, XVIII (October, 1929), 20–21.

Clement, J. A., and Boe, O. "Study in Secondary School Supervision Relative to Courses Offered in High Institutions of Learning," *Educational Administration and Supervision*, XVI (December, 1930), 705–13.

Cox, P. W. L. "Instruments of Creative Supervision," *High School Teacher*, V (April, 1929), 123–27.

Douglass, H. R. "High School Principal Supervises," *Minnesota Journal of Education*, XI (November, 1930), 101–2.

Gardiner, C. A. "Supervisory Work the Chief Function of the High School Principal," *American School Board Journal*, LXXIII (October, 1926), 56.

Graham, B. G. "Supervision of Classroom Teaching in the Junior High School," *National Education Association Addresses and Proceedings* (1926), pp. 772–77.

Gray, W. S. "Technique of Supervising High School Practice Teaching," *School Review*, XXVII (September, 1919), 512–22.

Haggard, W. W. "How Instruction Is Supervised in the Senior High School, Rockford, Illinois," *American School Board Journal*, LXXIII (November, 1926), 55.

Halliburton, J. R. "Classroom Observation and Supervision," *High School Teacher*, III (November, 1927), 367–69.

———. "High School Supervision," *ibid.* (October, 1927), pp. 305–6.

Harap, H. "Revision of the High School Curriculum," *Department of Secondary School Principals Bulletin*, XXV (March, 1929), 13–20.

Hawkes, F. P. "Supervision of Teaching in the Junior High School," *Journal of Educational Method*, V (September, 1925), 2–7.

Hughes, C. L. "High-School Principal as a Supervisor of Teaching," *American School Board Journal*, LXXVIII (March, 1929), 47–48.

Hughes, J. M., and Melby, E. O. *Supervision of Instruction in High School*, "Northwestern University Contributions to Education: School of Education Series," No. IV. Bloomington, Ill.: Public School Publishing Co., 1930.

Judd, C. H. "Can High School Supervision Be Made Scientific?" *National Education Association Addresses and Proceedings* (1928), pp. 733–42.

Klopp, W. J. "Plan for the Improvement of Teaching in the High School: Woodrow Wilson High School and Junior College, Long Beach, California," *School and Society*, XXIX (June, 1929), 705–6.

Koch, Harlan C. "Some Aspects of the Department Headship in Secondary Schools," *School Review*, XXXVIII (April, 1930), 263–75.

Lindquist, R. D. "Secondary School Principal as a Supervisor," *Department of Secondary School Principals Bulletin*, XXXV (March, 1931), 72–74.

Massey, H. N., and Others. "Study of the Status of the Subject Supervisor in the Junior and Senior High Schools of the Largest One Hundred Cities of the United States," *ibid.*, pp. 194–200.

Newlon, J. H. "Creative Supervision in High Schools," *American School Board Journal*, LXXVIII (June, 1929), 40.

Richardson, B. C. "Democracy and Service: A Discussion of Supervision of Instruction in Senior High Schools," *Department of Secondary School Principals Bulletin*, XXV (March, 1929), 133–42.

Seybold, Arthur M. "The Supervision That Teachers Need," *High School Teacher*, III (June, 1927), 218–19.

Stetson, F. L. "The Organization of Supervision in Small High Schools," *High School*, VIII (November, 1930), 36–42.

Thompson, H. G. "Principals' Conferences as an Aid to High School Supervision," *High School Quarterly*, XVIII (April, 1930), 144–47.

Weber, C. A. "The Supervision of Instruction in the Small High School," *American School Board Journal*, LXXXI (September, 1930), 42.

Whinnery, K. E. "Supervision of Instruction," *High School Teacher*, I (December, 1925), 367.

Young, R. N. "Fourteen Points of Supervision," *ibid.*, II (April, 1926), 133–34.

QUALIFICATIONS OF SUPERVISORS OF MATHEMATICS

Adair, C. S. "What Teachers Want in Supervision," *School and Society*, XXVII (March 3, 1928), 254–57.

Arndt, C. O. "A Program of Self-Improvement for the Supervisor," *High School Journal*, XIV (May, 1931), 249–59.

Bird, G. E. "Teacher's Estimates of Supervisors," *School and Society*, V (June 16, 1917), 717–20.

Bobbitt, Franklin. "Mistakes Often Made by Principals," *Elementary School Journal*, XX (January and February, 1920), 337–46 and 419–34.

Bush, M. G. "Elements Common to All Good Supervision," *Journal of Educational Method*, IX (May, 1930), 462–66.

Clark, R. C. "The Creative School Administrator," *American School Board Journal*, LXXX (August, 1930), 46.

Cook, S. "Teachers' Ideas of Helpful Supervision," *Educational Administration and Supervision*, IX (December, 1923), 554–57.

Department of Superintendence, National Education Association, "Training for Supervision," *Eighth Yearbook*, pp. 207–62.

Fowlkes, J. G. "What a Supervisor Should Know and May Do," *Nation's Schools*, III (March, 1929), 67–69.

Gist, A. S. "The Art of Supervision," *Journal of Educational Method*, V (January, 1926), 192–96.

Hayes, Fannie B. "Supervision from the Point of View of the Teacher," *School Review*, XXXIII (March, 1925), 220–26.

Hubbard, E. B. "What Teachers Expect of Supervisors," *Detroit Journal of Education*, III (May, 1923), 416–17.

Kelley, G. K. "Types of Supervisors I Have Known," *American School Board Journal*, LXVIII (June, 1924), 54.

McGinnis, W. C. "Self-Analysis for Supervisors," *Journal of Education*, CXI (January 6, 1930), 17.

Morrison, R. H. "Qualities Leading to Appointment as School Supervisors and Administrators," *Educational Administration and Supervision*, XII (November, 1926), 505–11.

Ritehour, F. C. "Desirable Traits in the Supervisor," *American School Board Journal*, LXXIX (October, 1929), 32.

Shannon, J. R. "An Analysis of High School Supervisory Notes," *Educational Administration and Supervision*, XI (January, 1928), 9–14.

———. *Personal and Social Traits Requisite for High Grade Teaching in Secondary Schools.* Terre Haute: State Normal Press, 1928.

Taylor, J. S. "Some Desirable Traits of the Supervisor," *Educational Administration and Supervision*, IX (January, 1923), 1–9.

Underwood, F. M. "A Self-Analysis for Supervisors," *Journal of Educational Method*, VIII (April, 1929), 417–18.

FUNCTIONS OF THE SUPERVISOR

Adair, C. S. "Supervision from the Standpoint of the Teacher," *School Life*, XIII (March, 1928), 137.

Allen, R. D. "Three Types of School Supervision," *Journal of Educational Method*, IX (May, 1930), 477–81

Aseltine, John. "The Duties of a Department Head in a Large City High School," *School Review*, XXXIX (April, 1931), 272–79.

Barr, A. S. "An Analysis of the Duties and Functions of Instructional Supervisors," *University of Wisconsin Bureau of Educational Research Bulletin*, VII (January, 1926), 19–21, 69–72, 118–19.

———. "The Policy Governing the Activities of the Supervisors," *ibid.*, pp. 10–11.

———. "Scientific Supervision," *Journal of Educational Method*, VI (January, 1927), 190–201.

Billins, M. I. "Supervision from the Standpoint of the Teacher," *Journal of Education*, LXXXIII (February 17, 1916), 173–74.

Board of Education, Atchison, Kansas. "Special Supervision," *Elementary School Journal*, XXI (October, 1920), 83–84.

Bobbitt, Franklin. "The Special Supervisor," *Chicago Schools Journal*, XI (December, 1928), 121–27.

Bowden, W. T. "Word to the Supervisor," *Industrial Education Magazine*, XXV (September, 1923), 59–60.

Brim, Orville G. "Changing and Conflicting Conceptions in Supervision," *Journal of Educational Method*, X (December, 1930), 131–40.

Brown, E. J. *Self-rating Scale for Supervisors, Supervisory Principals and Helping Teachers.* Milwaukee: Bruce Publishing Co., 1929.

Burton, W. H. "What the Teacher Has a Right To Expect from Supervision," *Educational Outlook*, V (November, 1930), 19–25.

Cadwallader, Dorothy K. "Report of a Supervisory Program," *Journal of Educational Method*, VII (March, 1928), 252–53.

Collings, Ellsworth. "The Meaning and Function of Creative Supervision," *ibid.*, IV (June, 1925), 404–9.

Cook, Selda, "Teachers' Ideas of Helpful Supervision," *Educational Administration and Supervision*, IX (December, 1923), 554–57.

Cranor, Katherine. "A Self-scoring Card for Supervisors as an Aid to Efficiency in School Work," *ibid.*, VII (February, 1921), 91–102.

Deffenbaugh, W. S. "Improvement of Teachers in Service," *Elementary School Journal*, XXV (January, 1925), 380–86.

Dorsey, S. M. "Some Suggestions Looking to Cooperation and Efficiency in the Functioning of Supervisors and Principals," *National Education Association* (1927), pp. 436–40.

Dunn, F. W. "What Is Instructional Supervision?" *ibid.* (1923), pp. 758–64.

Elmer, Maude V. "The Supervisor's Day at School," *Journal of Educational Method*, VII (September and October, 1927), 11–16.

Ferguson, H. A. "Educational Guidance as Supervision," *Department of Secondary School Principals Bulletin*, XXV (March, 1929), 112–22.

Garretson, O. K. "In-Service Training of Teachers in High Schools in Oklahoma," *School Review*, XXXIX (June, 1931), 449–60.

Giles, J. T. "Supervision of Instruction in Geometry," *High School Quarterly*, XVI (April, 1928), 177–82.

Ginsburg, M. B. "A Cooperative Supervisory Program," *Department of Elementary School Principals Bulletin*, X (April, 1931), 236–42.

Gist, A. S. "The Art of Supervision," *Journal of Educational Method*, V (January, 1926), 192–96.

Goodier, Floyd T. "Promoting the Growth of Teachers in Service," *American School Board Journal*, LXVI (May, 1923), 57.

Gosling, T. W. "Adjustment of the Duties of the Supervisor to Those of the Principal," *Elementary School Journal*, XXVI (September, 1925), 18–21.

Hart, M. C. "Supervision from the Standpoint of the Supervised," *School Review*, XXXVII (September, 1929), 537–40.

Hayes, F. B. "Supervision from the Point of View of the Teacher," *ibid.*, XXXIII (March, 1925), 220–26.

Hepner, W. R. "Stimulating the Growth of Teachers in Service," *American School Board Journal*, LXXIX (December, 1929), 57–58.

Hill, Sallie. "Defects of Supervision and Constructive Suggestions Thereon," *National Education Association*, LVII (1919), 506–9.

Hirschman, M. L. "Supervising the Beginning Teacher," *Baltimore Bulletin of Education*, IX (February, 1931), 126–30.

Hollinger, J. A. "What Should Teachers Expect from Supervision?" *Curriculum Study and Educational Research Bulletin*, III (March, 1929), 161–67.

Horrall, A. H. "Supervision—Its Weaknesses and Opportunities," *American School Board Journal*, LXXIV (March, 1927), 52.

Horst, W. "Planned Supervision," *School Executives Magazine*, XLIX (January, 1930), 228–29.

Hunkins, R. V. "Supervision Is Teaching," *School and Society*, XXVIII (December 8, 1928), 728–30.

Johnson, F. W. "The Department Head," *School Review*, XXXIII (September, 1925), 523–31.

Judd, C. H. "Examples of Scientific Procedure in Supervision," *Department of Secondary School Principals Year Book*, XIII (March, 1929), 31–40.

Klopp, W. J. "The Function of the Director of Teaching in the Secondary School," *High School Teacher*, V (April, 1929), 142.

Koch, H. C. "Is the Department Headship in Secondary Schools a Professional Myth?" *School Review*, XXXVIII (May, 1930), 336–49.

Kohlbrenner, B. J. "What Supervision Do Teachers Receive?" *Catholic Educational Review*, XXIX (March, 1931), 146–55.

Lyons, C. A. V. "What Should a Teacher Expect from Supervision?" *Curriculum Study and Educational Research Bulletin*, III (March, 1929), 168–70.

Maxwell, C. R. "Analysis of the Work of the Supervisor," *School and Society*, XVII (January 6, 1923), 28.

Mead, C. D. "The Supervisor's Job," *Journal of Educational Method*, IV (March, 1925), 270–72.

Miller, C. S. "Consideration of the Functions of Supervisory Officers," *Secretaries' Report of the Pennsylvania Education Congress* (1929), pp. 22–26.

Neal, Elma A. "What the Supervisor Can Do To Encourage Scientific Attitude in the Classroom," *Journal of Educational Method*, VI (May, 1927), 372–76.

Newlon, Jesse H. "Reorganizing City School Supervision," *ibid.*, II (June, 1923), 404–11.

Norton, M. A. "How Adequate Supervision Can Bring about Better Articulation of the Units of American Education," *ibid.*, X (April, 1931), 387–90.

Nyquist, F. V. "The Itinerary of the Special Supervisor," *Educational Administration and Supervision*, XV (February, 1929), 139–46.

Reavis, W. C. "The Duties of the Supervising Principal," *Elementary School Journal*, XIX (December, 1918), 279–84.

Saunders, M. O. "What the Teachers Want from the Principal in His Capacity as a Supervisor," *School Review*, XXXIII (October, 1925), 610–15.

Shannon, J. R. "Devices and Techniques of Supervision," *Teachers College Journal*, I (September, 1929), 27–29.

Spain, C. L. "A New Definition of the Functions of the Supervisor," *Elementary School Journal*, XXVI (March, 1926), 498–506.

Spencer, C. R. "The Demonstration Lesson as an Agency in Supervision," *ibid.*, April, 1926, pp. 619–26.

Stauffer, E. S. "What Principals Do When They Supervise," *Educational Research Bulletin* (Ohio State University), V (April, 1926), 167–71.

Storm, H. C. "Three Elements in Effective Supervision," *American School Board Journal*, LXVI (May, 1923), 58.

Wagner, C. A. "Some Types of Misconceived Supervision of Instruction," *ibid.*, pp. 37–38.

———. "Supervision at Work," *ibid.*, LXIII (September, 1921), 34.

Wegner, H. C. "A Practical Outline Governing the Principles of Supervision," *ibid.*, LXXX (March, 1930), 53–54.

Wilson, T. H. "How the Principal Can Guide Both Teacher and Pupil," *Nation's Schools*, V (April, 1930), 35–40.

Wright, C. O. "Supervision Individualized," *High School Teacher*, VII (February, 1931), 69–70.

PREPARATION OF TEACHERS OF MATHEMATICS

Bagley, William C. "The Ideal Preparation of Teachers of Secondary Mathematics from the Point of View of an Educationist," *Mathematics Teacher*, XXVI (May, 1933), 271–76.

Blank, Laura. "Professional Preparation and Growth in Secondary School Mathematics," *ibid.*, XXIII (November, 1930), 451–57.

Cairns, W. D. "The Training of Teachers of Mathematics with Special Reference to the Relation of Mathematics to Modern Thought," *ibid.*, XXIV (May, 1931), 269–76.

Hughes, J. M. "Suggested Study on the Relation of Training of Teachers in Content Subject Matter to the Effectiveness of Instruction," *Journal of Educational Research*, XIX (March, 1929), 228.

Freeman, J. R. "Preparation of Teachers of Mathematics for the Junior High Schools," *School Science and Mathematics*, XXIII (December, 1923), 842–52.

"Report of the Committee of the New England Association of Teachers of Mathematics on Secondary School Mathematics: Preparation of the Teachers of Mathematics in the Secondary Schools of Massachusetts," *Mathematics Teacher*, VIII (June, 1916), 210–18.

Slichter, Charles S. "The Self Training of a Teacher," *School Science and Mathematics*, XXXII (June, 1932), 598–604.

Tripp, M. O. "Collegiate Courses in Mathematics for Prospective High School Teachers," *School Science and Mathematics*, XXIII (May, 1923), 482–88.

VISITING THE TEACHER

Anderson, C. J., Barr, A. S., and Bush, Maybelle G. *Visiting the Teacher at Work.* New York: D. Appleton & Co., 1925.

Barr, A. S. "An Evaluation of Items To Observe in Class Room Supervision," *Journal of Educational Research,* XVIII (June, 1928), 53–65.

Buckingham, B. R. "Visiting the Classroom," *Educational Research Bulletin,* VIII (May 1, 1929), 185–90.

Clem, Orlie M. "Fundamental Elements in Junior High School Instruction," *Educational Method,* XI (June, 1932), 521–30.

Fish, L. F. "A Set of Standards for Use in Classroom Visitation," *Kansas Teacher,* XXXII (March, 1931), 16–17.

Gates, C. Ray. *The Management of Smaller Schools,* chap. i. New York: Houghton Mifflin Co., 1923.

Greene, F. W. "Class Room Visitation," *High School Journal,* XIII (December, 1930), 397–99.

Johnson, P. A., and Umstattd, J. G. "Class Room Difficulties of Beginning Teachers," *School Review,* XL (November, 1932), 682–86.

Knudson, Charles W. *Evaluation and Improvement of Teaching.* Garden City, N.Y.: Doubleday, Doran & Co., 1932.

Koch, Harlan C. "Is the Department Headship in Secondary Schools a Professional Myth?" *School Review,* XXXVIII (May, 1930), 336–48.

Nietz, J. A. "What the Supervisor Looks For in Class Room Supervision," *American School Board Journal,* LXXXI (August, 1930), 41–42.

Power, Leonard. "How To Make Visits Profitable to Teachers," *Bulletin of the Department of Elementary School Principals,* VIII (April, 1929), 359–63.

Pratt, O. C. "The Technique of Visitation and Conference with Teachers," *American School Board Journal,* LXXX (May, 1930), 49–50.

Rugg, G. "Visitation as a Means of Diagnosis," *Department of Elementary School Principals Bulletin,* VIII (April, 1929), 355–59.

Snyder, L. G. "Class Room Visits and the Individual Conference," *Bulletin of the Department of Elementary School Principals,* X (April, 1931), 258–61.

Thorne, N. C. "Technique of Class Room Visitation," *High School,* VIII (November, 1930), 43–48.

Yawberg, A. G. "Instructional Supervision with the Announced Visit as an Important Factor," *School Review,* XXXI (December, 1923), 763–76.

DEPARTMENT MEETINGS

Allen, T. T. "Teachers Meetings upon a Democratic Basis," *Educational Administration and Supervision,* V (January, 1919), 19–24.

Barr and Burton. *The Supervision of Instruction.* New York: D. Appleton & Co., 1926.

Boettner, Emma A. "Plans for Teachers' Meetings," *Journal of Educational Method,* I (October, 1921), 19–25.

Boyce, Arthur C. "Methods for Measuring Teachers' Efficiency," *Fourteenth Yearbook: The National Society for the Study of Education,* Part II. Chicago: University of Chicago Press, 1915.

Clark, R. C. "The Teachers' Meeting," *Education,* XLVIII (December, 1927), 242–48.

Hawkes, F. P. "Supervision of Teaching in the Junior High School," *Journal of Educational Method,* V (September, 1925), 2–7.

McLaughlin, H. P. "A Plan for Meetings of Mathematics Teachers in a High School," *Mathematics Teacher,* XXII (November, 1929), 418–26.

Saul, E. L. "Professional Teachers' Meetings for the High School," *School Review,* XXX (May, 1922), 371–77.

Schutte, T. J. *Schutte Rating Scale for Teachers.* Yonkers, N.Y.: World Book Co., 1923.

Southall, Macie. "Vitalized Group Study and Faculty Meetings," *Journal of the National Educational Association,* XVIII (1929), 315.

Thomas, John S. "Encouraging Discussion in Teachers' Meetings," *Elementary School Journal,* XXX (February, 1930), 444–49.

Woltring, Clara. "Increasing the Productivity of Demonstration Lessons for Student Teachers," *Educational Administration and Supervision,* XVI (January, 1930), 12–18.

RATING OF TEACHING

Adams, W. C. T. "Two Teacher Rating Cards," *American School Board Journal,* LIX (December, 1919), 30.

Alexander, Carter. "Standard Tests as an Aid in Supervision," *ibid.*, LIV (January, 1917), 17–18.

Almy, H. C., and Sorenson, Herbert. *Rating Scale for Teachers.* Bloomington, Ill.: Public School Publishing Co., 1931.

———. "A Teacher Rating Scale of Determined Reliability and Validity," *Educational Administration and Supervision,* XVI (March, 1930), 179–86.

Baker, B. K. "Objective Measurement of Teacher-Traits," *American School Board Journal,* LXXV (December, 1927), 43.

Boyce, A. C. "Methods for Measuring Teachers' Efficiency," *Fourth Yearbook: National Society for the Study of Education,* Part II (1915).

Bradford, Mary D. "How the Superintendent Judges the Value of a Teacher," *American School Board Journal,* LIV (January, 1917), 19–20.

Breckenridge, William E. "Judging a Teacher of Mathematics," *Mathematics Teacher*, VIII (March, 1916; June, 1916), 151–59, 169–72.

Brueckner, L. J. *Judgment Test of Teaching Skill.* Minneapolis: Educational Test Bureau, 1932.

Clark, R. C. "A Scale for Measuring Teachers," *American School Board Journal*, LXII (February, 1921), 39–40.

Clem, Orlie M. "Fundamental Elements in Junior High School Instruction," *Educational Method*, XI (June, 1932), 521–30.

Committee of the Chicago Principals' Club. "A Study of the Facts That Characterize Superior Teaching," *Third Yearbook* (Chicago, June, 1928), pp. 141–86.

Connor, William. "A New Method of Rating Teachers," *Journal of Educational Research*, X (May, 1920), 338–58.

Crabtree, J. W. "Rating of Teachers," *National Education Association Proceedings*, LIII (1915), 1165–67.

Evans, R. O. "Standards for Judging Good Teaching," *American School Board Journal*, LXXX (June, 1930), 40.

Gray, W. S. "Rating Scales, Self Analysis, and the Improvement of Teaching," *School Review*, XXIX (January, 1921), 49–57.

Hammond, G. B. "Teacher Rating as an Administrative Function," *Bulletin of the Department of Elementary School Principals*, IX (April, 1930), 533–38.

Franzen, Carl G. F. "Improvement Sheet for Algebra," *School Science and Mathematics*, XXXII (December, 1932), 939–43.

Hill, C. W. "The Efficiency Ratings of Teachers," *Elementary School Journal*, XXI (February, 1921), 438–43.

Knudson, Charles W. *Evaluation and Improvement of Teaching.* Garden City, N.Y.: Doubleday, Doran & Co., 1932.

Landsittel, F. C. "A Score Card Method of Teacher-Rating," *Educational Administration and Supervision*, IV (June, 1918), 297–304.

Miller, Wm. A. "Teacher Rating from the Principal's Point of View," *American School Board Journal*, LXXVIII (May, 1929), 48.

Monroe, Walter S. "Observable Characteristics of Efficiency in Teaching," *Elementary School Journal*, XXVII (April, 1927), 597–600.

Moore, J. G. "The Rating of Teachers," *American School Board Journal*, LXXV (December, 1927), 48.

Moss, F. A., Hunt T., and Wallace, F. C. *Teaching Aptitude Test*, "George Washington University Series" (1927).

Peik, W. E. *Objective Analysis of Recitations.* Minneapolis: Educational Test Bureau, 1932.

Rugg, H. O. "Self Improvement of Teachers through Self-rating: A New Scale for Rating Teachers' Efficiency," *Elementary School Journal,* XX (May, 1920), 670–84.

Scudder, J. W. "Problem of Teacher Rating," *Journal of Educational Method,* VII (May, 1928), 367–68.

Somer, G. T. *Pedagogical Prognosis,* "Columbia University Contributions to Education," No. 140 (1923).

Wagner, Charles. "A Plan of Procedure for the Inauguration of a Teacher Rating Scale and a Related Salary Schedule," *American School Board Journal,* LXVII (July, 1923), 38–39.

Weber, S. E. "Rating Teachers and Principals To Improve Their Service," *ibid.,* LXXX (April, 1930), 47–49.

CHAPTER II

A PROGRAM OF DEPARTMENTAL TESTING AND RESEARCH

Planning the testing program for the department.—The testing movement has produced a variety of tests that are useful as instruments of measuring and of improving the results of teaching. Intelligence tests, prognostic tests, and mathematical-ability tests aid the teacher in pre-estimating the pupil's success in the study of the various mathematical subjects and in grouping pupils for the purpose of simplifying the teaching problems arising from individual differences. Inventory tests disclose the pupil's former experiences in the course or unit he is about to study. They clarify the teaching situations and show the needs of individuals and of groups of pupils. Practice and instructional tests enable the pupil to test himself and to determine his achievement and understandings as he proceeds with the study of a unit. Unit tests give evidence of the pupil's understanding of processes and principles when he has finished the units of a course. Standardized achievement tests measure the achievement of pupils at certain periods, e.g., at the end of a semester. They make it possible to compare classes and schools with each other on the basis of scientifically derived norms. Tests are used to carry on research. They replace opinion by facts and reveal reliable information which may be used to improve instruction and supervision. Weaknesses in the course may be located to point the way to the development of better methods and better teaching devices.

Formerly the major purpose of testing was to secure evidence for determining the grades of pupils. The foregoing list of tests makes it apparent that this is no longer the case. The written test material which the teacher may study is more valuable than the grade. He may gather from it much about the learning process of the pupil, and he will find in it definite suggestions either for remedial steps to be taken immediately or for improvements to be made when he teaches the course again. The time spent on such testing

is well employed. It no longer causes a break in instruction but has become one of its essential features.

Complete descriptions of the various types of tests that may be used, the purposes of each type, and the methods of constructing tests are found in chapters i and vii of *The Technique of Teaching Mathematics in Secondary Schools.* The aim of the present chapter is to show how tests may be used by teachers and supervisors to improve the work of the department. Standardized tests have a place in this program. They are convenient and necessary. However, many tests will have to be devised by the teachers themselves to meet situations for which no standardized tests are available.

Since teachers are not always sufficiently trained, the task of actually constructing the tests is often delegated to an expert, e.g., the supervisor or an instructor in a college department of mathematics or education. Usually this is not satisfactory. A person not actually teaching a course is likely to overestimate or underrate what the pupils are able to do. His vocabulary differs from that of the teachers. This leads to confusion and dissatisfaction on the part of teachers and pupils. They consider the tests unfair. Often a wrong attitude toward testing develops. Hence, the plan of training teachers to make tests is recommended to a department that enters seriously upon a campaign of improving instruction.

The training program might begin with a thorough study of the characteristics of good tests and of the values and purposes of the various testing devices employed in modern standardized and unstandardized tests. The teachers offering a given course should assume the responsibility for constructing the tests for that course. As soon as a first tentative draft of a test is made, it should be submitted to the supervisor or to an expert on tests for corrections, suggestions, criticisms, and approval. They form the basis for revising the test. When this has been done, a typed copy and several good carbons should be made. An excellent way to try out the revised test is to let each teacher giving the course actually take the test. Usually this suggests further changes, especially in the arrangement and statements of the test items. It is a good plan to try the test also on several pupils who have previously taken the course.

Mimeographed copies should then be made and the test is ready

to be administered to the pupils taking the course. The teachers themselves should score the tests and during the scoring keep a record of all facts which may be of use in a future revision.

It is evident that the foregoing plan involves a great deal of labor, but the results will repay the teachers for the work. It may take several years to accumulate a satisfactory set of tests for a course. However, when the job is finished, the tests will be as useful and reliable as standardized tests. They may be used over and over, which will compensate the teachers for the time and effort spent in constructing the tests.

The supervisor of the department should supply the leadership in the testing program. He should therefore be a person to whom the teachers may look for guidance and often for assistance. He must be familiar with the technique of testing and with the literature relating to the subject.

The following is an illustration of a test made by the teachers of an eighth-grade course to measure the results on a unit on problems to be solved by means of formulas.[1] It will be seen that the test is far more comprehensive than is usually the case with tests administered at the end of a chapter. Thirty-eight test items are given and five different abilities are tested.

UNIT V—PROBLEMS SOLVED BY FORMULAS

Pupil...Date..........................

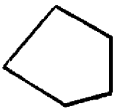
Teacher...

A. *Meanings of mathematical terms.*—The items of the left column illustrate the words of the right column. In the blank space following each word of the right column, write the *number* of the item in the left column that illustrates it *best.*

1. Perimeter formula...

2. $c=\pi d$ Quadrilateral...

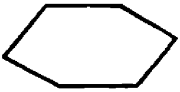
3. $d=rt.$ Uniform motion formula......................................

4.  Hexagon...

[1] E. R. Breslich, *Eighth-Year Mathematics*, chap. viii.

5. $p=\frac{r}{100}\times b$ — Pentagon..

6. $i=\frac{prt}{100}$ — Interest formula..

7.  — Circumference formula..

8. $p=a+b+c+d$ — Percentage formula..

B. *Making a formula.*

1. The price of oranges is 40 cents a dozen.
   a) Find the price of 2, 3, 4, 5 . . . . 10 dozen.
   b) Arrange these facts in tabular form.
   c) Make a formula for finding the price *p* of *n* dozen oranges.
2. Make a formula representing the relationship in the table

No. of articles	1	2	3	4	5	6
Prices in dollars	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3

. Formula:..

C. *Exercises in per cents.*—Express as per cents:

1. $\frac{1}{5}=$..............................per cent
2. $\frac{3}{4}=$..............................per cent
3. $.65=$..............................per cent
4. $1.25=$..............................per cent

Express as common fractions:

5. 25 per cent=..............................
6. 60 per cent=..............................
7. $5\frac{1}{2}$ per cent=..............................
8. 130 per cent=..............................

Express as decimals:

9. 16 per cent=..............................
10. 50 per cent=..............................
11. $37\frac{1}{2}$ per cent=..............................
12. $8\frac{1}{3}$ per cent=..............................

D. Solve the following problems by means of formulas:

1. A man saved 20 per cent of his income of $4,500. How much did he save?
2. In this class there are 29 pupils; 17 are boys. What per cent are boys?
3. A salesman received a commission of $90 on a sale. This was a 4 per cent commission. What was the amount of the sale?
4. A $55 suit was reduced to $35. What per cent was the discount?
5. Find the interest on a note of $400 at 6 per cent for 90 days.
6. How long will it take to earn $360 interest on $1,000 invested at 6 per cent?
7. How much money must be invested at 4 per cent to earn $500 a year interest?
8. What is the radius of a circle whose circumference is 628 feet?

9. An automobile traveled 100 miles in 2½ hours. At what rate did it travel?

E. *Functional relationships.*—Complete the following sentences:
1. The perimeter of a regular pentagon depends on..............................
....
2. The percentage depends on..............................and..............................
....
3. The interest received on a given sum of money depends on..............................
..............................and..............................
4. The gain made on a sale depends on..............................and
....
5. The circumference of a circle depends on..............................
....

The scoring of the tests.—Since improvement of instruction is the major purpose of testing, the papers should be read and scored by the teachers even if they have been scored previously by an assistant. Much valuable information will be obtained. Errors should be noted which should be discussed with the class or with individual pupils. Faulty procedures will be found which need to be corrected. A record should be made of the deficiencies which show the need of additional teaching and practice. Slow and weak pupils should be identified.

Few activities that teachers may carry on are more profitable than the studying and analyzing of the written work done by the pupils.

Keeping the class record.—When the papers have been scored, the results should be carefully tabulated on a class record sheet similar to the one illustrated in Table I. It gives the results of an arithmetic test which consists of ten parts. The first eight parts test arithmetical computation; the remaining two, problem-solving. A separate record sheet should be made for each class. It should be supplemented by a statement giving a detailed diagnosis.

Diagnosis.—The totals at the bottom of Table I show the parts of the test in which the entire class is weak or deficient. They will form the basis for planning reteaching of the principles and processes which the parts represent.

The column of totals at the right shows the range of abilities in the class and identifies the slow and the good pupils. The best may

be excused from reteaching to spend their time more profitably in the library or on some supplementary project. The papers of the slow pupils should be studied with special care. A separate record should be made for each pupil to serve as a basis for individual conferences with the teacher. The record should be organized to give

TABLE I

CLASS RECORD SHEET FOR THE REAVIS-BRESLICH ARITHMETIC TESTS

Course............................School...Grade..................
Class period.....................Teacher..Date....................

Parts of Test / Class Enrolment	Computation									Problem-Solving			Totals
	I	II	III	IV	V	VI	VII	VIII	I–VIII	IX	X	IX and X	
1.......	3	7	6	4	6	2	11	6	45	6	12	18	63
2.......	6	7	7	4	7	5	10	6	52	9	12	21	73
3.......	3	5	5	2	4	3	11	1	34	9	13	22	56
4.......	4	7	5	3	9	4	11	4	47	12	15	27	74
5.......	8	7	12	7	8	12	11	4	69	11	18	29	98
6.......	5	5	5	4	7	7	11	9	53	8	15	23	76
21.......	6	6	6	4	12	10	7	4	55	11	13	24	79
22.......	12	10	11	4	5	14	11	7	74	3	9	12	86
23.......	3	6	7	2	6	9	10	9	52	6	0	6	58
24.......	2	5	7	3	4	3	11	0	35	12	12	24	59
Totals right.	148	171	157	91	169	178	282	139	327	228	308	536	1,863
Medians..	5.3	7	6.2	4.1	6.3	7.3	10.2	5.7	52	9.3	13	21	73
Averages.	5.6	6.3	5.8	3.4	6.2	6.6	10.5	5.2	49.2	8.5	11.4	19.8	69

information about the pupil's achievement on the test as a whole and also on the separate parts. It should disclose facts like the following.

The test shows that Pupil A succeeds better (not as well as) in computation than in problem-solving.

Pupil B is deficient in both arithmetical computation and problem-solving.

Pupil C is fairly accurate, but too slow to accomplish much in the given time. He needs practice.

Pupil D is very inaccurate and works too rapidly.

Pupil E is slow and inaccurate.

Pupil F has difficulty with the zero, as in $4\times0=4$.

Pupil G is not sure of the fundamental combinations, as in $2\times3=5$.

Pupil H makes mistakes in carrying in addition.

Pupil I forgets that he has borrowed in subtraction.

Pupil J subtracts fractions by subtracting numerator from numerator and denominator from denominator, as in $\frac{3}{4}-\frac{2}{7}=\frac{1}{3}$.

Pupil K fails to reduce the results to the simplest form.

Pupil L does not understand the process at all.

Pupil M adds numerator to numerator and denominator to denominator as in $\frac{1}{8}+\frac{1}{4}=\frac{2}{12}$.

Pupil N uses the wrong operation, as in $\frac{3}{4}-\frac{1}{2}=\frac{5}{4}$, or $\frac{1}{4}\times\frac{1}{5}=\frac{2}{9}$; or $\frac{7}{10}+\frac{5}{4}=\frac{28}{50}$.

Pupil O uses cross-multiplication, as in $\frac{4}{7}\times\frac{2}{3}=\frac{12}{14}$.

Pupil P confuses dividend and divisor, as in $\frac{4}{7}\div\frac{2}{3}=\frac{7}{4}\times\frac{2}{3}$.

Pupil Q does not know where to place the decimal point in division.

The foregoing list could be extended. However, it contains a sufficient number of illustrations to show the character of the diagnosis. In this case it discloses the seriousness of arithmetical difficulties encountered by pupils in the secondary school and the urgent need for corrective treatment.

Remedial instruction.—Reteaching should be provided in three ways: reteaching of the entire class, of small groups, and of single individuals.

If the record sheet discloses weaknesses and deficiencies of the class with the exception, perhaps, of a few outstanding pupils, reteaching of the class should come first. Care should be taken to teach general processes and principles rather than specific terms or problems taken from the test. Teaching should be followed by practice in which test items may be used together with other practice material. The class should then be tested again. All of the class reteaching and testing should be done during regular class periods.

If the diagnosis of the pupils' papers shows small groups of pupils deficient in certain processes, it is economical to reteach them

in groups. Thus, one group may study the addition of fractions, another the placing of the decimal point in division, and another the solving of verbal problems. This work should not be done during the regular class period but after school hours or some time during the school day if this can be arranged.

Usually there are a few pupils who require more teaching than is being provided in the foregoing groups. They need individual instruction. Corrective work with slow pupils is tedious, time consuming, and often a thankless job. However, it is a responsibility which the teacher and the department cannot evade.

With the pupil's test paper before them, teacher and pupil discuss errors and their causes, and try to find ways of eliminating them. Sometimes a clumsy procedure holds the pupil back. Thus he may be adding a column of figures saying: 2 and 4 are 6 and 5 are 11, etc., rather than 2, 6, 11, etc. His difficulty might be traced to lack of knowledge of the fundamental combinations. Lack of study habits, confidence, or practice might be the cause. A physical handicap often interferes with successful performance. When the cause of difficulty is discovered, it is usually a simple matter to remove it and to secure immediate improvement.

The use of standardized tests in measuring achievement in mathematics.—The procedure will be illustrated with an arithmetic test designed for secondary-school pupils. However, an algebra or geometry test would serve the purpose just as well. It is being more and more recognized that arithmetical skill should be listed among the important objectives of the secondary school. When the pupil has completed the elementary school, he has been taught all of the arithmetical processes and principles which people in general need in their everyday work. Arithmetic thus ceases to be taught as a school subject. Nevertheless, in the mathematics of the secondary school and in some of the other subjects a thorough knowledge of arithmetic is essential. Hence, instruction in arithmetic must be supplied by the teachers of such subjects whenever weaknesses appear. This is true particularly of courses in mathematics and in the sciences. Teachers in all courses must assume the responsibility of helping pupils when they are found deficient in arithmetic.

The fact that several investigations have shown that a pupil's success in high-school algebra does not depend on a successful study

of arithmetic should not be interpreted to mean that arithmetic is no longer essential. It merely points out that arithmetic and algebra differ in various important characteristics.

To insure continuous growth in arithmetic until arithmetical maturity is reached the teacher should secure objective evidence by testing pupils at regular intervals, not less frequently than once each year. The results should be followed up with a remedial program such as will be outlined in the next pages. It should be understood that this is the least that may be done. Further opportunities for discovering deficiencies in arithmetic to be followed by corrective treatment will present themselves frequently in all mathematics courses. They should not be overlooked or slighted. Similarly, to insure continuous growth in algebra the teacher should test pupils at regular intervals and carry on corrective work whenever algebraic deficiencies are discovered.

In any grade the pupils show marked individual differences. Thus it is not uncommon to find in a seventh-grade class some pupils who have only fifth-grade arithmetical ability and others who surpass ninth-grade pupils. Hence, remedial work in many cases should be an individual matter which requires testing and diagnosing in addition to that carried on with the class as a whole. The first step in the program is therefore to select and administer a test which does not go too much into detail, which can be given in a relatively brief time, and which classifies pupils with a fair degree of accuracy. The next step will be to test in greater detail those pupils who are found at the lower end of the scale. The diagnosis of the test papers will indicate specific remedial measures.

The administration of tests.—When a standardized test is given to all classes in a department, the question arises whether it is best to have one person administer the test or to have each teacher test his own classes. The first method secures uniformity but has the disadvantage that in most schools it is not possible to have the testing done by one person. Furthermore, classroom conditions are not as usual if a stranger comes in to do the testing. This might easily cause tension or even excitement, which may have a bad effect on the work of the pupils.

Teachers should be trained to administer standardized tests. It is an easy matter to obtain such training because definite instruc-

tions for giving tests are usually supplied by the publishers. All that is necessary is to learn to follow directions to the letter, to watch the time limits carefully, not to omit anything and not to add any explanations. A good way to get acquainted with a test is to take it. Practice in giving the test may be obtained by administering it to one or several pupils before attempting it with the entire class. A department meeting may profitably be devoted to a study and discussion of the test. The following are some of the points that should be kept in mind by the teacher who administers the test:

The teacher should not announce the test in advance.

The teacher's manner and voice should be as usual.

The pupils, preferably those in the first row, should distribute the booklets to the class. This leaves the teacher free to give directions and to see that all pupils are ready to begin.

During the class period the teacher should supervise the room carefully but not give the impression that he mistrusts the pupils.

Scoring the test.—In all cases the tests should be scored by the teachers rather than by assistants, otherwise much information of importance to improvement of teaching will be lost.

Class records.—A record sheet similar to that illustrated on page 48 should be made out for each class. At the bottom of the sheet a row containing the norms should be added.

Diagnosis.—As described on page 47, the totals on the record sheet supply information as to deficiencies of the entire class and individual pupils. A careful record should be made, which will form the basis for corrective work to be done with both the class and the individuals in need of special instruction.

Graphical representation.—Further information as to the arithmetical ability of a class may be secured from graphical representation of the test results. Hence, the next step is to construct histograms. For the foregoing test it is advisable to make a histogram for Parts I–VIII to represent ability in computation, one for Parts IX and X to represent ability in problem-solving, and one for the entire test.

If departmental records are kept, or if various teachers of a department wish to make comparisons of classes, they should come to definite agreements as to the type of graphs to be made, the

units to be employed, the legends to be written, and other details. Uniformity in these matters will facilitate the comparisons and add greatly to the usefulness of the graphical work. Figure 1 is a sample of graphical representation. It shows the results on Part X for each of four seventh-grade classes and for all four classes combined. Separate graphs were originally made by four different teachers. Later they were placed together as shown, and a graph for the entire group was then added by the supervisor.

Before the graphs are made, certain details should be worked out jointly in a departmental conference. They should be strictly observed by the teachers when they make the graphs. The teachers of the four classes agreed on the following details:

1. Metric paper should be used in the graphical work.
2. The origin is to be placed on the vertical zero line.
3. The unit on the vertical axis should be 1 cm. = 2.
4. The unit on the horizontal axis should be determined from the range. Thus since for Part X the scores varied from 3 to 19, the horizontal scale should be 1 cm. = 2 and should be numbered from 3 upward.
5. The legends should read uniformly, as:

ARITHMETICAL ACHIEVEMENT, TEST X: PROBLEMS WITH NUMBERS

Course....................	Class period....................	Number of pupils..........
Teacher....................	Class median....................	
	Grade median....................	
	Norm....................	

6. The norm 7.25 should be marked on the horizontal axis. A line should then be drawn through it at right angles to the axis. Another vertical line should be drawn through the 7.5 point to show the grade median. A third vertical line should show the median for each class.

To enable the reader to grasp readily the meanings of the three lines in each histogram the grade median may be shown by a red line and the norm and class medians by black lines, respectively solid and dotted. In Figure 1 the norm is marked by the heavy vertical line and the class medians by dotted lines.

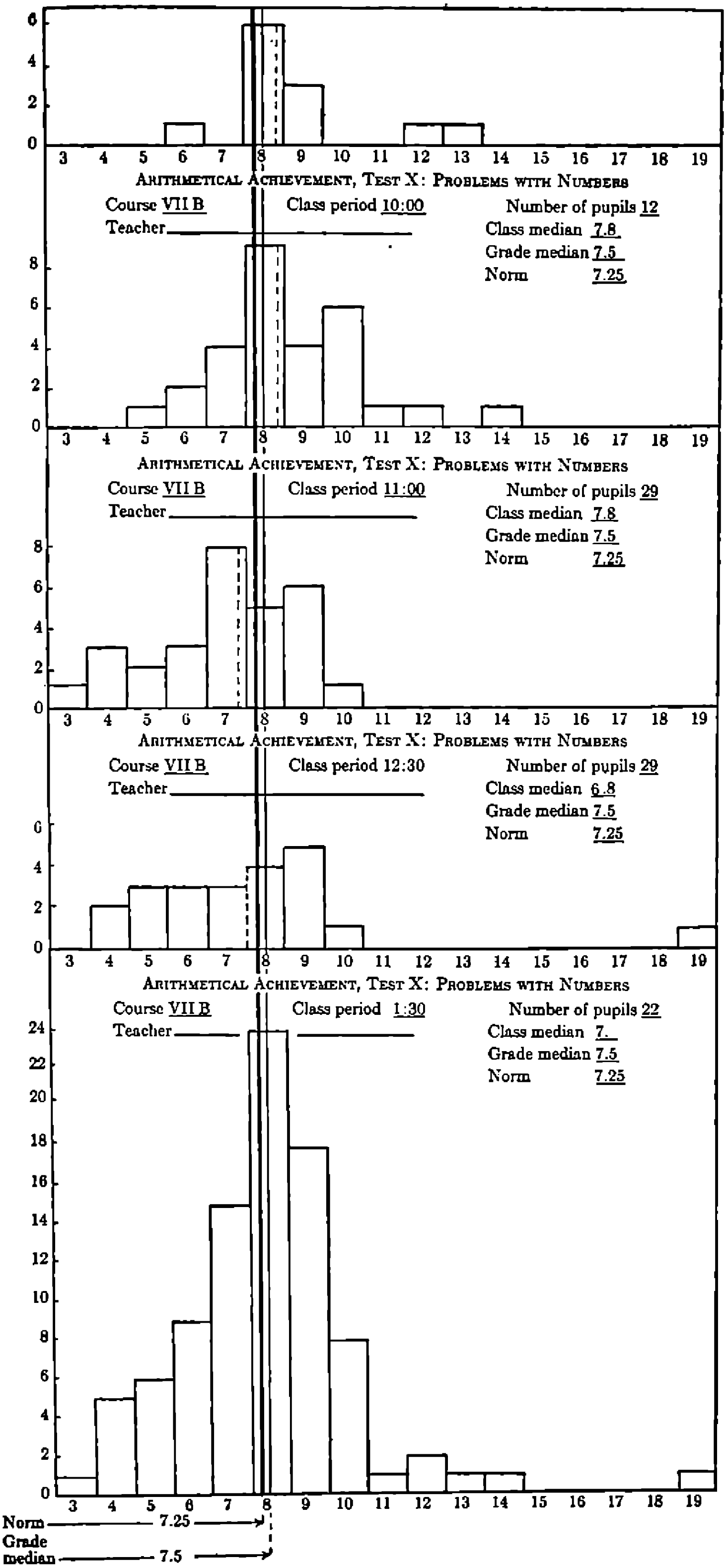


FIG. 1. Arithmetical Achievement Test X: Problems with Numbers, Course VII B. Entire group: Number of pupils, 92; grade median, 7.5; norm, 7.25.

Diagnosis.—The histograms (Fig. 1) show that two of the sections and the group as a whole slightly surpass the norms, and that two sections fall a little below the norm, which suggests the need of remedial work for the pupils who fall below the norm. Many of them lack mastery of processes which are of unquestionable importance. Evidently work in arithmetic will have to be provided for them until arithmetical maturity is reached. It is not expecting too much to bring the slow pupils up to the norm for their respective grades.

One pupil in the group solved 19 of the 20 problems in 4 minutes. Fourteen pupils reached or surpassed the norm for the tenth grade, and 2 surpassed that of the twelfth grade. Twelve pupils fell below the sixth-grade norm and 1 barely reached that of the fifth grade. The range from 3 to 19 of correct solutions shows that it would be unwise and unnecessary to subject the whole group to the same type of remedial teaching and practice.

Remedial treatment.—Studies of arithmetical achievement usually disclose three types of arithmetical deficiency among pupils of the secondary schools: lack of understanding of the concepts of the subject, lack of proficiency in the fundamental processes, and lack of ability in problem-solving. It should be a social objective of every school to develop arithmetical maturity of all pupils before they finish the high-school course.

Several procedures have been advocated to accomplish this objective.

1. *Daily drill in arithmetic.*—A few minutes are assigned each day for drill. The method is generally effective in raising the average achievement of a class. The objections are that the bright pupil wastes time; that the slow pupil is rushed and profits but little; and that the results attained are not lasting.

2. *Formal review of arithmetic at the beginning of first-year algebra.*—The objections stated in 1 apply here. Furthermore, pupils dislike to begin secondary-school mathematics with arithmetic.

3. *Elimination of typical errors.*—Numerous studies are available which list such errors and difficulties as are commonly experienced by pupils. Corrective classes are formed for pupils who are found to commit these errors.

4. *Constant attention to arithmetical difficulties.*—Teachers in all classes pay continuous attention to accuracy and neatness in arith-

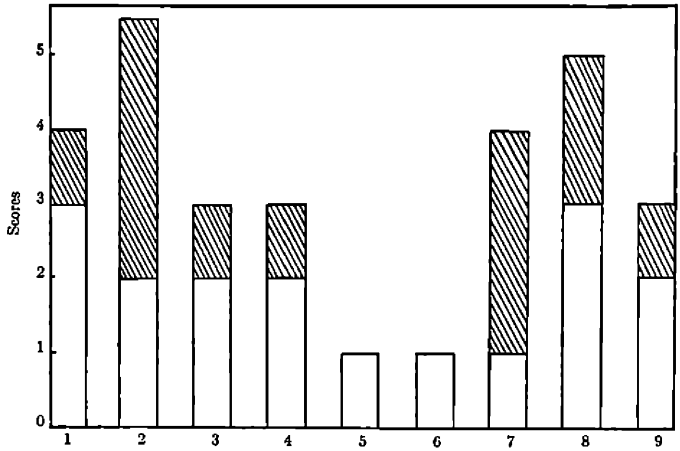


FIG. 2a.—Improvement of nine pupils in Part I, addition, due to reteaching.

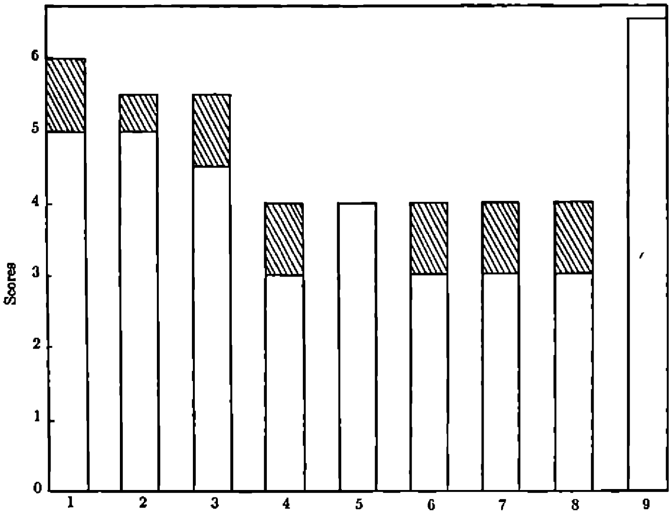


FIG. 2b.—Improvement of nine pupils in Part II, subtraction, due to reteaching.

metic in all courses. Reviews are given whenever in the course arithmetical deficiencies are discovered. Thus arithmetic is placed on the same basis with other high-school courses.

The fourth method has been very satisfactory. Most of the work is done in the classroom, but when individuals need additional teaching they receive special attention.

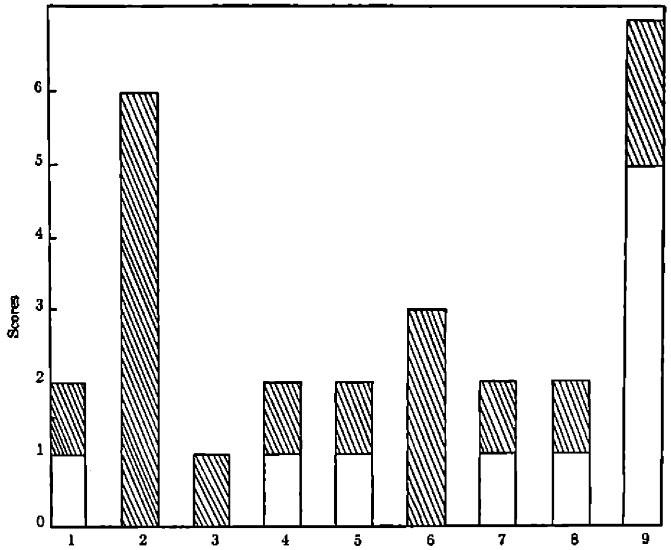


FIG. 2c.—Improvement of nine pupils in Part III, multiplication, due to reteaching.

The class record sheet will show that some of the processes should be retaught immediately. On others reteaching may be deferred until a place in the course is reached where proficiency is necessary. If the test has been given previously, the gains or losses on the test and on each of the parts should be carefully noted. They will throw light on the extent to which the course contributes, or fails to contribute, to arithmetical growth. For the parts in which losses have occurred, provision for practice should be made in various places of the course to eliminate such losses in the future.

Corrective work with individual pupils brings results slowly. In some cases it may take weeks before the efforts of the teacher are rewarded. Figure 2 shows the improvement of nine pupils on Parts

I, II, and III (addition, subtraction, and multiplication) of the test. They came to the instructor for special help during a period of several weeks; received individual assistance; worked practice ex-

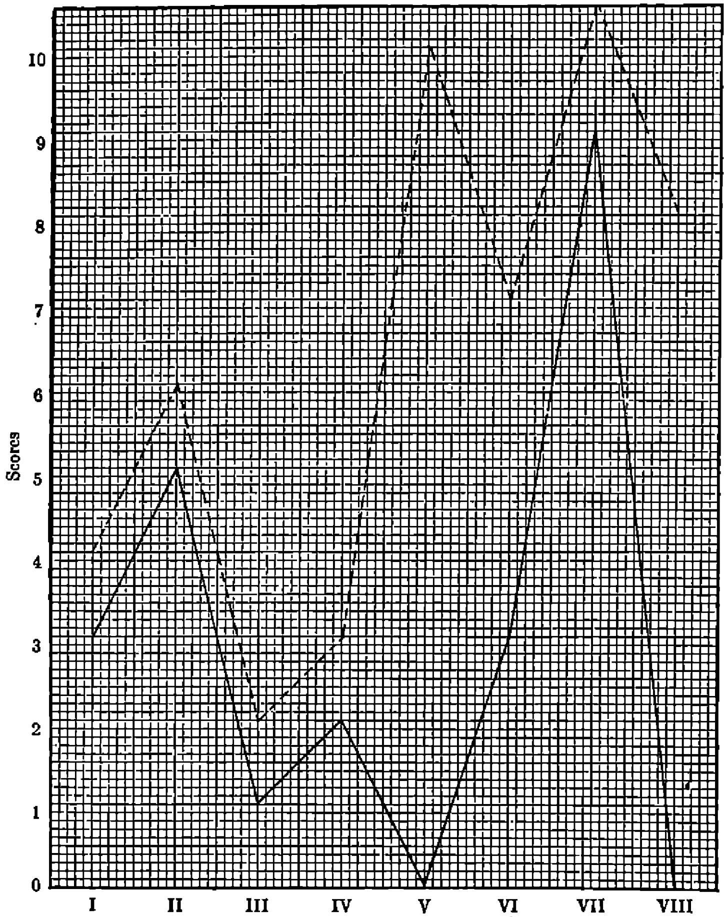


FIG. 3.—Improvement of Pupil 1 on Parts I–VIII.

——— Original scores

------ Scores on retest

ercises which were marked by the teacher and returned; and did many problems on the blackboard which were criticized by the instructor. At the end of the period the test was taken again with the results shown in the diagram. All pupils improved in Part III

(multiplication). Pupil 5 did not improve in Parts I and II (addition and subtraction). Pupil 6 failed to improve in Part I and pupil 9 in Part II.

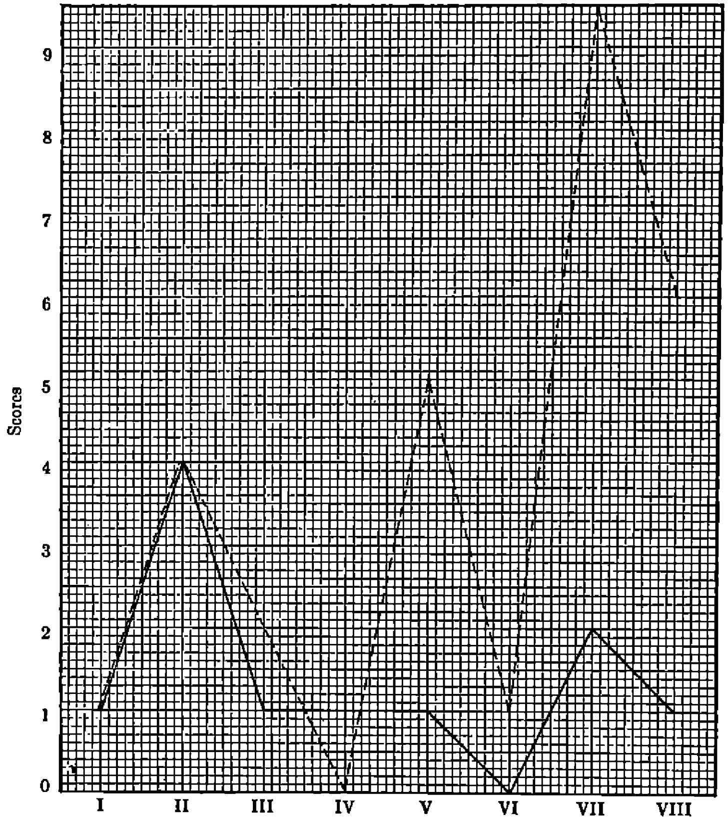


FIG. 4.—Improvement and loss of Pupil 5 on Parts I-VIII
——— Original scores
------ Scores on retest

Figure 3 shows the record of Pupil 1 on Parts I–VIII. It represents the kind of progress to be expected. Figure 4 shows the record of Pupil 5, who did not gain in Parts I and II and registered a loss in Part IV (long division). Figure 5 is the record of Pupil 6, who made no gains in Parts I and VII (addition and subtraction of fractions).

Summary.—In the foregoing pages the use of a standardized test has been described which involves the following features:

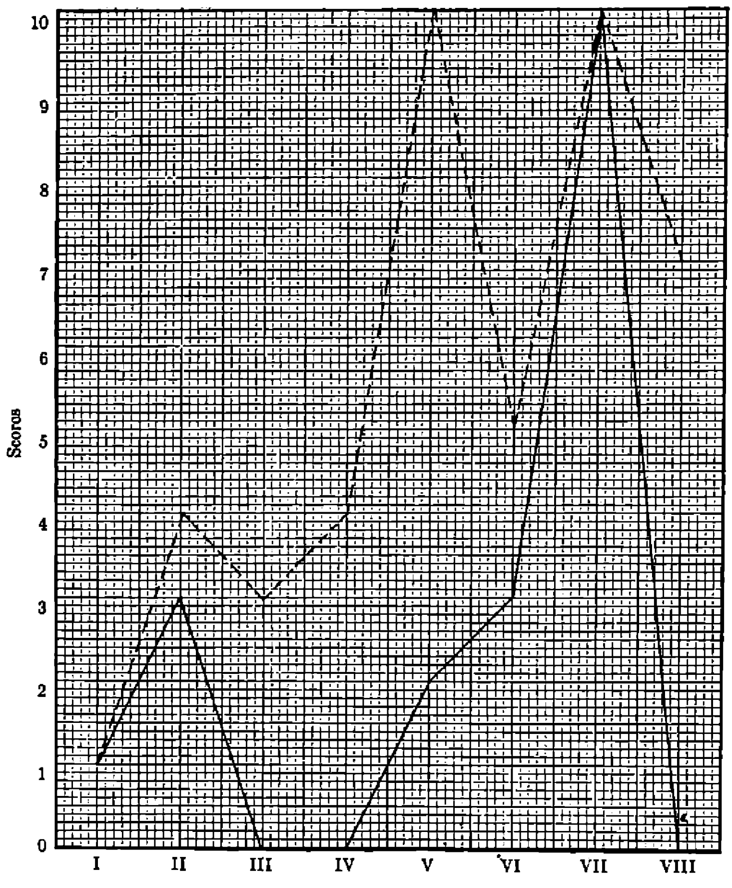


FIG. 5.—Improvement of Pupil 6 on Parts I–VIII

——— Original scores

------ Scores on retest

A. Teachers should train themselves to administer standardized tests.

B. The tests should be scored by the teachers themselves. Otherwise most valuable information needed to bring about improvement in teaching will be lost.

C. A complete record of each class should be kept. The following outline was suggested:

I. Record of class........... Grade...........
   Period........... Date...........
   a) A class record sheet should be filled out. Diagnosis of the record sheet should discuss
      1. The medians at the bottom
      2. The totals at the right
      3. Unusual pupils
         (a) Bright pupils
         (b) Slow pupils
   b) A record should be made of individual errors and difficulties
   c) The facts should be represented graphically
      1. Graphs should follow definite directions as to axes, units, legends, medians, and norms
      2. Diagnoses of histograms should compare class medians and norms, state range of scores, and show the remedial work needed
   d) Remedial instruction should be provided
      1. Reteaching of entire class
      2. Reteaching of small groups having specific difficulties
      3. Reteaching of individuals
      4. Tabular and graphical record of improvements of pupils in each part and on the entire test

The supervisor's study of the tests.—If the teachers keep careful records of the results attained by their classes, it is not a difficult matter for the supervisor or for some member of the department to summarize all the findings and to secure additional information of interest and importance to the department. The purpose is not to evaluate the teaching ability of the various teachers but to identify weaknesses among classes and to take steps to bring about improvement. Thus, it was shown in Figure 1 how the sections of one grade may be compared with each other, with the norms, and with the grade median. The important conclusion to be drawn from the fact that two of the classes fell below the norm while the other two surpassed them is that the teachers of the first two classes have to face a more serious problem in bringing about improvement.

The foregoing outline of a testing program for arithmetic applies equally to algebra and geometry. Mathematical growth during the

entire secondary period should be carefully studied by every department. Definite evidence should be secured as to progress in all subjects. This requires departmental testing at regular intervals.

The method of interpreting the results will be again illustrated with the arithmetic test. Figure 6 shows the arithmetical achievement of an entire school. It is evident that more attention needs to be given in the school to computational arithmetic than to problem-solving. The teachers of all grades except IX should outline a program for developing greater proficiency in computing.

Furthermore, since the tests were taken at the beginning of the year, the teachers of Grades VIII and X should be interested. They should do what they can to strengthen the work in arithmetic in these grades in order that their pupils reach the next grade better prepared in arithmetic.

To make the corrective work definite and intelligent, data for the various parts of the test should be secured for all grades. Such data are shown in Figure 7 for Parts I–IV. In Part I (addition) Grades IX and XII have not made the progress that comparatively should be expected of them, although it is possible that Grade XII has reached the maximum achievement. In Part II Grades VII and IX need to be strengthened. In Part III attention should be given to Grades VII, VIII, and X, and in Part IV to Grades VII, IX, and X. Similar information may be gathered from a graphical representation of each of the remaining parts of the test.

Comparing the mathematics departments of several schools.—Comparison of the achievements of one department with those of other schools of the same type helps the supervisor to identify weak and strong phases of instruction. It stimulates the desire to improve where results are unsatisfactory and gives encouragement where they are satisfactory. It illustrates the seriousness and difficulty of the problems of instruction where most or all schools show weakness.

Tables II–VIII give the results for a standardized algebra test[2] administered to classes of sixteen different schools. They enable the supervisor in each of these schools to draw some valuable conclusions for his teachers. Each table represents the data for one of the six parts of the test, except the last, which summarizes the results

[2] *Breslich Algebra Survey Test* (second semester), Form A.

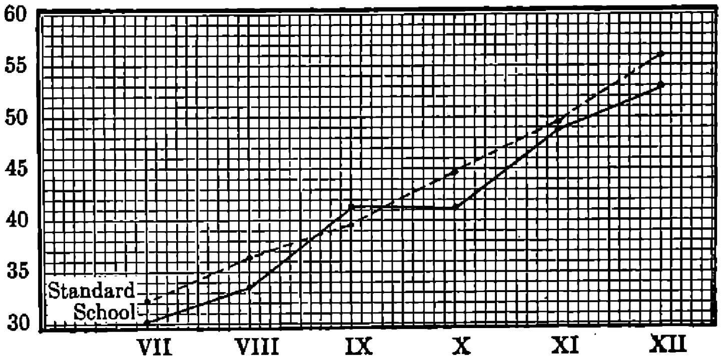


FIG. 6a.—Parts I-VIII, computation, Grades VII-XII

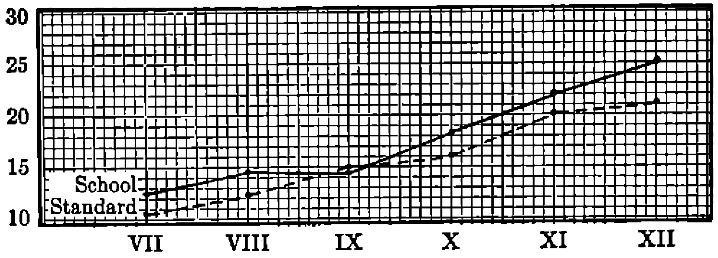


FIG. 6b.—Parts IX and X, problem-solving, Grades VII-XII

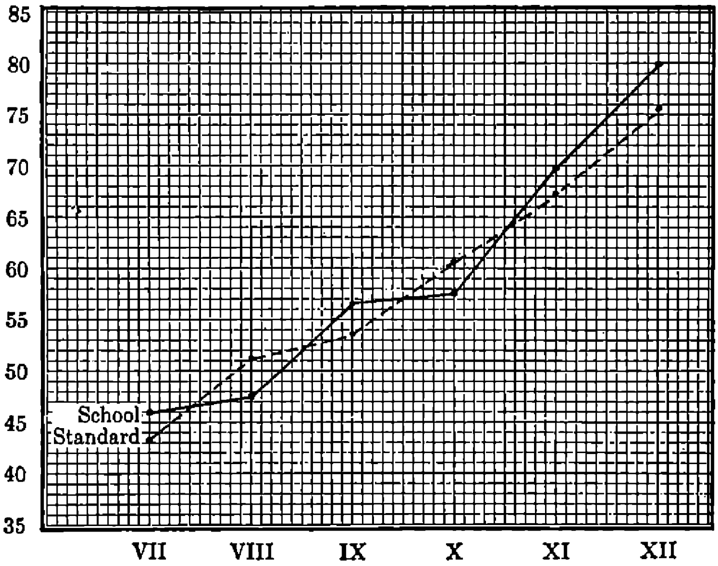


FIG. 6c.—Parts I-X, arithmetical achievement, Grades VII-XII

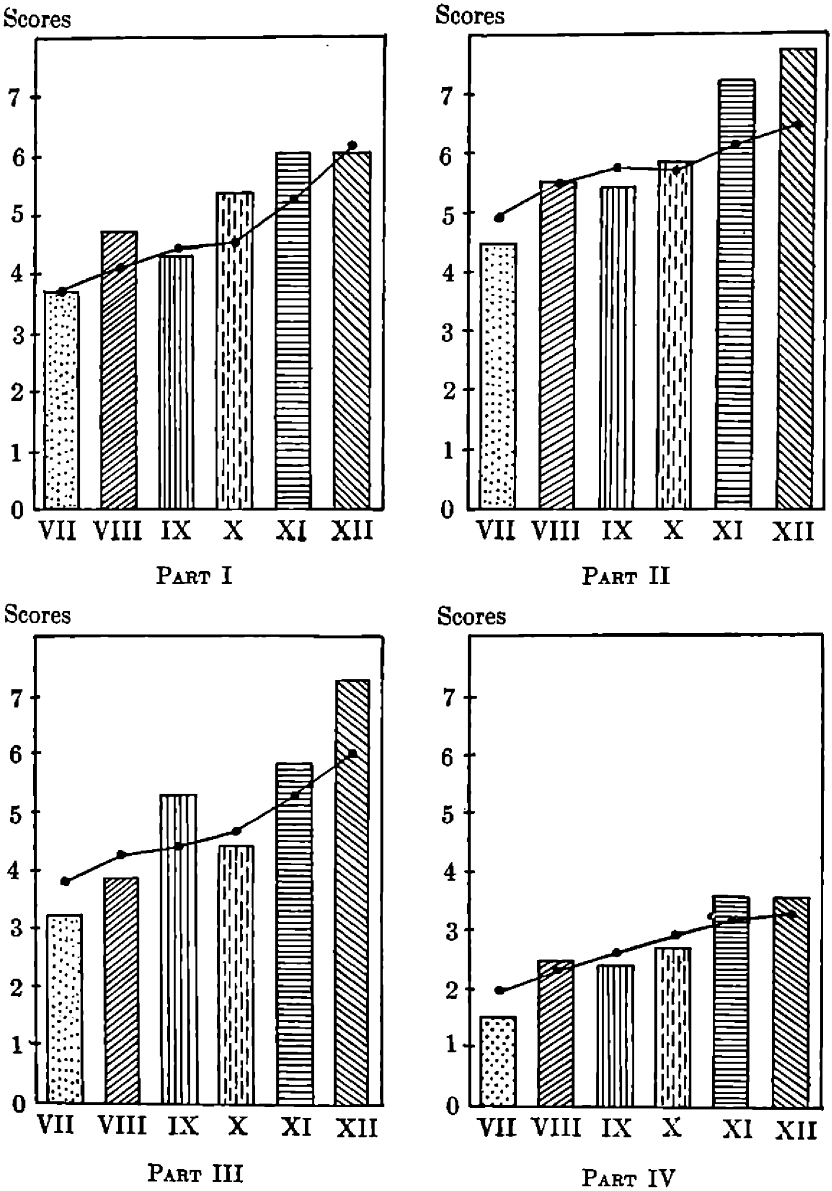


Fig. 7.—Comparison of median scores, Parts I–IV, Grades VII–XII, with the norms for these grades.

on the entire test. The following questions may be raised by the teacher in regard to Table II, which measures the pupils' knowledge of algebraic concepts.

1. Does the distribution for the entire group show that pupils have learned the meaning of these concepts?
2. Is the distribution for my school similar to that for the entire group?
3. How does the median of my group of pupils compare with the norm?
4. How does the range of scores for my group compare with that of the entire group?
5. Which pupils should be exempt from reteaching?
6. Which pupils should receive intensive reteaching?

Apparently the group as a whole has acquired a fair comprehension of algebraic concepts. The distribution of School 1 is very similar to that of the group, and Schools 5 and 15 show but little resemblance to it. No pupils in any of the schools missed all of the test items, and in four schools there were pupils who responded correctly to all of the questions. The best results were attained in School 15, and the poorest in Schools 3 and 16. Even in the best school some individuals are greatly in need of reteaching.

Table III shows a fairly normal distribution for the test on the fundamental algebraic processes. The distributions of Schools 6 and 10 show resemblance to that of the entire group. The results of School 12 are very poor, while School 11 makes a very good showing. Schools 6, 7, 11, and 15 have some strong pupils, and Schools 12 and 13 have a full share of poor material. In the entire group there are 57 pupils who could not do a single problem correctly, and a large number of pupils is greatly in need of further teaching.

Part III of the test measures ability to solve equations. The results are shown in Table IV. As in the first two parts, the distribution for the entire group indicates that the subject is receiving its share of attention in the schools. Forty-seven pupils, however, were unable to solve a single equation. School 6 contributes one-third of this number. The poorest showing is made by School 16 and the best by Schools 7, 8, 10, and 11.

The distribution for the group in Table V is not nearly as satisfactory as for the first three parts of the test. It appears that problem-solving ability is not as thoroughly attained as proficiency in

TABLE II

Breslich Algebra Survey Test (Second Semester, June, 1932), Form A

PART I. ALGEBRAIC CONCEPTS

Score	Norms	Schools 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
17	....	....	....	....	....	1	....	2	....	....	....	....	....	....	3	8	....	14
16	....	1	1	....	12	....	15	27	11	....	11	11	2	....	1	26	1	119
15	....	7	3	1	9	....	7	27	11	1	13	25	2	3	5	17	....	131
14	....	4	....	....	12	2	19	31	17	1	16	12	4	2	7	20	....	147
13	....	14	3	3	13	3	17	42	16	10	20	21	6	2	7	13	1	191
12	....	8	5	....	19	7	29	52	19	7	27	22	5	3	17	2	....	222
11	....	11	7	1	16	5	34	48	20	7	15	28	7	6	17	10	2	234
10	....	21	9	4	24	4	34	47	31	8	17	31	9	3	11	6	4	263
9	....	21	4	3	28	8	25	59	13	15	13	20	12	3	11	5	4	244
8	....	15	9	7	24	7	35	42	24	18	13	17	8	4	4	6	7	240
7	....	23	7	5	14	8	21	41	21	20	14	20	8	5	....	4	10	221
6	....	22	7	4	12	3	21	36	13	17	8	15	7	6	3	6	4	184
5	....	16	6	7	9	2	9	15	10	10	5	6	10	3	1	3	8	120
4	....	8	3	2	2	8	12	10	5	11	5	3	1	5	1	1	6	83
3	....	5	4	3	2	4	1	3	1	4	5	5	1	2	1	2	3	46
2	....	3	....	4	1	2	1	....	2	3	1	1	....	....	....	....	1	19
1	....	1	3	1	....	3	2	....	....	1	....	1	....	1	....	....	....	13
0	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
Total	....	180	71	45	197	67	282	482	214	133	183	238	82	48	89	129	51	2,491
Median	9.3	8.8	8.6	7.3	10.7	8.5	10.4	10.7	10.6	8.0	11.7	11.0	9.5	8.5	11.7	14.3	7.4	10.3
25 per cent	7.2	6.5	6.3	5.2	8.4	5.0	8.1	8.4	8.1	6.3	8.6	8.5	7.2	6.2	10.1	10.9	5.3	7.7
75 per cent	11.7	11.0	11.2	9.3	12.8	11.3	12.6	13.2	13.1	10.1	13.7	13.5	11.8	11.4	13.1	16.1	8.9	12.9

TABLE III

Breslich Algebra Survey Test (Second Semester, June, 1932), Form A

PART II. ALGEBRAIC PROCESSES

Score	Norms	Schools																Total
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
13	......	2	2	....	4	1	6	8	6	....	5	12	....	....	....	12	....	58
12	......	7	4	....	2	....	20	36	15	....	8	14	....	1	....	13	....	120
11	......	9	2	....	8	2	25	34	15	....	17	19	1	....	4	8	1	145
10	......	7	9	1	5	1	32	48	25	3	20	17	1	....	2	5	....	176
9	......	14	2	1	11	4	19	58	16	1	15	24	....	1	4	10	....	180
8	......	13	7	3	22	6	31	68	33	7	18	22	2	2	8	13	....	255
7	......	20	5	3	17	5	30	54	15	12	23	25	6	3	12	9	1	240
6	......	17	5	9	16	4	34	49	30	17	16	27	4	1	15	9	5	258
5	......	18	3	13	21	5	23	42	20	18	17	23	6	5	13	12	1	240
4	......	13	5	1	27	9	20	46	16	23	17	28	6	5	13	13	10	252
3	......	23	7	5	25	12	25	22	11	26	14	15	14	12	7	10	14	242
2	......	16	6	7	18	7	11	13	8	18	8	11	18	5	6	6	11	169
1	......	15	7	2	16	3	2	3	1	5	4	1	19	10	4	4	3	99
0	......	6	7	....	5	8	4	1	3	3	1	....	5	3	1	5	5	57
Total	......	180	71	45	197	67	282	482	214	133	183	238	82	48	89	129	51	2,491
Median	5.4	5.9	6.0	5.6	5.4	4.4	7.7	8.2	8.1	4.6	7.6	7.6	2.9	3.6	6.0	7.6	3.5	6.7
25 per cent	3.6	3.3	2.6	3.5	3.4	2.8	5.4	5.8	5.7	3.3	5.1	5.2	1.8	1.9	4.3	4.6	2.4	4.2
75 per cent	8.4	8.5	9.6	6.6	8.1	7.5	10.4	10.1	10.3	6.4	10.2	10.1	4.9	5.2	7.6	10.1	4.5	9.3

TABLE IV

Breslich Algebra Survey Test (Second Semester, June, 1932), Form A

PART III. SOLVING EQUATIONS

Score	Norms	Schools 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
34	....	....	....	....	2	....	....	1	....	....	2	1	....	....	....	....	....	6
33	....	....	....	....	....	....	....	....	....	....	3	....	....	....	....	....	....	3
32	....	....	1	....	1	....	1	20	5	....	3	10	....	1	....	7	....	49
31	....	....	1	....	1	....	....	....	....	....	....	....	....	....	1	....	....	3
30	....	....	....	1	1	....	....	11	7	2	7	5	....	....	....	4	....	38
29	....	....	2	1	3	....	2	25	9	2	1	12	....	....	....	4	....	61
28	....	2	5	1	1	....	....	12	6	2	8	3	....	1	1	5	....	47
27	....	....	....	1	1	....	....	30	8	3	12	11	....	1	1	3	....	71
26	....	2	2	2	3	....	4	27	5	2	12	10	1	....	2	5	....	77
25	....	1	2	....	4	....	4	27	12	2	8	7	1	1	4	3	....	76
24	....	2	....	2	3	....	5	29	11	6	6	15	....	....	....	4	....	83
23	....	6	2	....	7	....	4	22	7	6	7	9	1	....	5	2	....	78
22	....	6	3	1	4	1	8	13	12	8	11	5	2	....	6	6	....	86
21	....	9	1	1	4	....	3	32	13	4	5	7	1	....	5	3	....	88
20	....	8	2	....	4	2	8	21	15	3	7	15	1	2	2	4	1	95
19	....	3	3	1	3	....	9	21	9	11	9	10	3	....	6	4	....	92
18	....	12	1	1	15	7	6	24	9	4	6	10	4	1	5	6	....	111
17	....	6	3	....	7	1	14	18	8	5	13	15	1	1	4	3	....	99
16	....	8	3	2	13	2	11	16	11	5	9	9	....	1	5	7	1	103
15	....	5	1	2	13	2	8	15	12	10	9	12	....	3	2	3	1	98
14	....	17	2	3	6	3	18	17	11	7	10	7	3	1	5	9	2	121
13	....	10	2	1	11	4	14	19	11	4	11	7	9	7	5	1	2	118
12	....	8	3	....	12	4	9	12	6	7	3	20	4	3	3	10	3	107
11	....	6	5	3	10	6	21	10	2	9	4	3	8	1	6	4	3	101
10	....	16	4	1	10	6	7	12	5	4	2	9	4	4	1	....	2	87

TABLE IV—*Continued*

Score	Norms	Schools 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
9	......	8	2	1	7	3	14	9	6	4	6	5	3	6	....	6	4	84
8	......	9	1	2	12	6	26	11	5	8	2	6	8	3	8	4	5	116
7	......	10	4	1	7	3	9	10	1	4	1	3	4	2	2	2	..	63
6	......	10	2	5	10	2	24	4	2	4	1	6	9	2	1	7	7	96
5	......	5	3	5	5	5	20	5	3	2	3	3	6	2	4	3	4	78
4	......	4	....	....	1	1	....	3	....	....	1	....	....	1	1	....	1	13
3	......	3	5	3	11	2	17	4	....	2	1	3	6	1	3	7	7	75
2	......	....	1	3	3	1	1	2	1	2	....	....	....	....	1	....	2	18
1	......	....	1	....	....	....	....	....	2	....	....	....	....	....	....	....	....	3
0	......	4	4	1	2	6	15	....	....	1	....	....	3	2	....	3	6	47
Total	......	180	71	45	197	67	282	482	214	133	183	238	82	48	89	129	51	2,491
Median	15.0	13.7	13.3	11.2	13.8	10.8	11.4	21.3	20.2	15.9	20.1	19.1	11.0	11.0	16.5	16.8	6.8	16.2
25 per cent	10.2	9.0	7.4	5.9	8.9	6.9	6.7	15.2	14.8	11.3	15.1	13.2	6.6	8.3	11.2	11.6	3.7	10.3
75 per cent	20.3	18.5	22.1	18.8	18.5	14.4	16.8	26.2	24.9	21.9	26.2	25.0	13.7	15.0	21.6	24.7	11.1	22.6

TABLE V

Breslich Algebra Survey Test (Second Semester, June, 1932), Form A

PART IV. PROBLEMS

Scores	Norms	Schools 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
18			1		4							1				1		7
17																		
16							1	2			1	8				5		17
15								3										3
14			2		3		9	13	1		1	7				7		43
13								3										3
12		4	7		19		9	20	1		8	7	4	2	1	12		94
11								1										1
10		6	10		20		18	37	19		21	11	8	9	2	9		170
9									1									1
8		11	9	4	28		24	63	14	4	37	20	9	6	2	10		241
7														3				3
6		26	9	12	32	7	47	101	31	12	43	29	14	6	21	13	3	406
5								1										1
4		60	21	9	40	9	95	90	55	30	39	71	21	9	37	31	13	630
3								1										1
2		42	7	14	31	23	50	97	58	59	21	65	18	7	19	13	7	531
1									2		1						1	4
0		31	5	6	20	28	29	50	32	28	11	19	8	6	7	28	27	335
Total		180	71	45	197	67	282	482	214	133	183	238	82	48	89	129	51	2,491
Median	4.5	4.3	6.3	4.3	6.2	2.2	4.7	6.0	4.3	2.7	6.5	4.7	4.7	6.3	4.5	4.8	0.9	4.6
25 per cent	2.6	2.3	4.3	2.4	2.9	0.6	2.8	2.7	2.3	2.1	4.3	2.6	2.7	2.9	2.8	2.3	0.5	2.5
75 per cent	6.9	6.1	10.2	6.4	8.9	4.0	6.8	8.3	6.4	4.4	8.6	6.8	8.1	8.8	6.2	10.2	4.3	6.9

* "Very little done with this type of problem."

Breslich Algebra Survey Test (SECOND SEMESTER, JUNE, 1932), FORM A

PART V. FRACTIONS

Scores	Norms	Schools 1	2	3	4	5	6	7	8	9	10	11	12	13	*14	15	16	Total
27					1		2	7			3	2				2		17
26																		
25																		
24		5			2		8	14	7	1	6	2				5	1	51
23																		
22																		
21		7	3		4		14	22	8	1	8	21		2		8		98
20																		
19																		
18		11		1	11	1	26	34	16	3	18	25		1		15		162
17																		
16																		
15		25	4	1	9	1	28	86	33	4	20	27		4	2	13	4	261
14																		
13											4							4
12		20	2	2	23	3	45	96	25	17	31	41		7	6	12	4	334
11																		
10								2										2
9		32	12	4	22	8	42	79	29	23	35	25	1	4	5	13	5	339
8								2							1			3
7									1									1
6		33	12	6	43	12	41	78	34	29	30	49	4	8	29	11	8	427
5												1						1
4								3										3
3		32	17	9	46	11	44	40	36	34	16	32	11	13	20	22	15	398
2								3					1					4
1																		
0		15	21	22	36	30	32	16	25	21	12	13	65	9	26	28	14	385
Total		180	71	45	197	66	282	482	214	133	183	238	82	48	89	129	51	2,490
Median	6.4	9.3	3.9	3.1	6.4	3.3	9.6	12.2	9.4	6.4	9.9	10.0	0.6	6.3	3.9	9.5	3.8	9.1
25 per cent	3.8	3.9	0.8	0.5	3.3	0.6	3.7	6.8	3.8	3.4	6.6	6.3	0.3	3.2	0.9	3.2	0.9	3.6
75 per cent	9.8	15.1	9.3	6.5	12.0	6.7	15.3	15.5	15.3	9.7	15.5	15.6	0.9	12.3	6.7	15.8	9.3	12.9

* "Not taught."

TABLE VII

Breslich Algebra Survey Test (Second Semester, June, 1932), Form A

PART VI. FUNCTIONAL RELATIONSHIPS

Scores	Norms	Schools																Total
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
31	.....	....	....	....	2	....	....	3	....	....	2	....	....	2	....	....	....	9
30	.....	....	....	....	....	....	3	3	1	....	1	1	....	....	....	1	....	10
29	.....	....	1	....	2	....	1	13	....	....	1	2	....	1	....	3	....	24
28	.....	1	....	....	....	....	1	5	1	1	3	5	1	2	2	1	....	23
27	.....	1	....	....	2	....	3	11	8	2	4	6	1	3	3	1	1	46
26	.....	....	1	....	3	....	3	12	10	1	5	6	....	2	6	5	1	55
25	.....	2	3	....	7	2	7	15	8	3	6	10	2	2	6	4	1	78
24	.....	5	5	3	4	....	10	17	6	5	4	6	1	1	7	6	....	80
23	.....	6	5	1	7	....	13	28	6	7	15	10	3	1	8	3	....	113
22	.....	7	3	....	12	1	6	26	8	1	4	11	6	3	6	7	....	101
21	.....	4	3	1	10	2	12	20	14	6	4	11	2	6	6	3	5	109
20	.....	10	5	2	13	1	11	24	18	6	6	16	9	....	7	9	1	138
19	.....	13	5	2	9	1	14	33	5	4	3	17	3	3	7	8	1	128
18	.....	7	3	2	11	1	13	35	19	4	4	20	2	4	5	9	4	143
17	.....	10	1	5	8	1	16	26	19	5	9	6	4	4	5	5	....	124
16	.....	11	2	1	15	2	23	38	18	13	6	21	3	4	4	7	7	175
15	.....	9	4	2	9	3	15	23	14	4	6	13	5	3	1	8	2	121
14	.....	10	3	1	10	5	26	21	12	4	9	17	4	....	7	6	2	137
13	.....	4	3	1	11	2	11	16	8	4	9	15	4	2	1	4	5	100
12	.....	5	6	1	10	2	13	26	9	8	5	11	6	1	....	11	5	119
11	.....	10	3	2	15	5	18	13	6	5	4	8	4	2	3	5	4	107
10	.....	12	1	2	9	8	9	6	6	5	7	7	1	....	....	5	4	82
9	.....	9	....	4	6	2	15	11	6	5	6	8	3	1	1	8	2	87
8	.....	7	2	5	10	2	13	7	4	6	3	2	3	....	....	6	2	72
7	.....	12	2	2	3	8	10	19	1	3	6	4	2	1	....	2	1	76
6	.....	5	6	1	4	4	5	13	2	11	7	4	3	....	1	1	1	68
5	.....	7	2	....	1	7	6	11	2	4	9	1	1	....	2	1	1	55

TABLE VII—*Continued*

Scores	Norms	Schools 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
4	......	3	1	2	1	1	1	2	2	7	7	....	1	....	....	....	....	28
3	......	6	....	1	2	2	2	4	1	3	8	....	1	....	....	....	....	30
2	......	1	1	3	....	1	2	....	....	2	8	....	2	....	....	....	....	20
1	......	2	....	1	1	1	....	1	....	4	7	....	2	....	1	....	....	20
0	......	1	....	....	....	2	....	....	....	....	5	....	3	....	....	....	....	11
Total	......	180	71	45	197	66	282	482	214	133	183	238	82	48	89	129	50	2,489
Median	16.6	14.6	16.8	11.8	16.4	10.4	15.7	18.1	17.8	13.9	14.1	18.1	15.2	19.7	20.9	17.1	14.0	16.6
25 per cent	12.6	9.1	11.9	8.3	11.8	6.6	11.4	13.5	14.5	7.8	6.3	14.0	9.8	16.5	17.3	12.4	11.4	11.7
75 per cent	21.2	19.3	22.1	18.0	21.0	14.5	20.0	22.5	21.6	19.7	21.8	21.8	20.2	25.0	24.3	21.6	18.4	21.2

the manipulative processes. When 339 pupils cannot solve a single problem and 532 others solve only one, the conclusions may be drawn that the teaching of verbal problems is a difficult task and that the procedure needs to be greatly improved. This is particularly true of Schools 5 and 16, in which about one-half of all the pupils failed to solve a single problem.

The results for Part V of the test, Table VI, are as distressingly poor as for Part IV. The method of teaching fractions must be improved. In School 12 it was found that 65 out of 82 pupils could not do correctly a single problem on fractions. The results in Schools 3 and 5 are almost as bad. School 7 makes a very good showing.

Table VII presents some interesting facts. The total distribution indicates that functional thinking is being acquired by pupils of first-year algebra to the same extent as the basic manipulative processes. The function concept seems to be receiving considerable attention in teaching, as is suggested by the uniformity of the distributions of all of the schools.

Table VIII enables the supervisor to compare his school with what the group has accomplished on the entire test. It seems that School 7, which has the largest number of pupils, makes the best record, and Schools 5 and 16 rank lowest in the group.

The use of tests in departmental research.—It has been shown in the foregoing pages that tests may be used for a variety of purposes. When given at the end of the unit they identify the principles on which classes and individuals need further teaching. They form the basis for remedial teaching and aid in the improvement of instruction. They enable the teachers and supervisors to obtain a continuous record of achievement by which classes taking the same subject may be compared with each other and with classes of former years. They give evidence of the extent to which the objectives of the teaching of mathematics are being attained by the pupils. When the results of the tests are summarized by the supervisor they may be used to guide the teachers to greater efficiency. Testing may thus become a co-operative enterprise in which all teachers are interested because the purposes are fully understood by them.

A testing program leads naturally to research and to experimen-

TABLE VIII

Total Scores: *Breslich Algebra Survey Test* (Second Semester, June, 1932), Form A

Scores	Norms	Schools 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
140–43	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
135–39	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
130–34	....	....	....	....	....	....	....	....	....	....	....	1	....	....	....	....	....	1
125–29	....	....	....	....	2	....	....	1	....	....	1	....	....	....	....	1	....	5
120–24	....	....	1	....	2	....	2	5	....	....	....	4	....	....	....	2	....	16
115–19	....	....	....	....	1	....	1	6	1	....	2	9	....	....	....	5	....	25
110–14	....	....	1	....	1	....	1	8	3	....	2	4	....	1	....	4	....	25
105–09	....	1	1	....	1	....	3	11	7	....	3	5	....	....	....	5	....	37
100–104	....	1	....	....	2	....	6	15	5	....	5	4	....	1	1	8	....	48
95–99	....	3	2	....	5	....	10	21	8	....	14	6	....	1	2	7	....	79
90–94	....	3	4	....	5	1	9	24	9	1	8	10	1	1	2	6	....	84
85–89	....	8	2	2	6	1	16	38	14	2	7	11	2	2	4	5	1	121
80–84	....	4	8	3	16	....	5	31	14	4	17	15	1	1	2	4	....	125
75–79	....	10	5	2	9	....	14	57	21	7	15	17	3	2	8	8	1	179
70–74	....	9	1	....	6	....	12	41	28	4	17	13	....	3	4	6	2	146
65–69	....	11	4	2	18	2	23	52	10	10	12	23	2	2	14	4	....	189
60–64	....	18	4	....	15	7	21	40	16	18	15	22	4	3	13	10	1	207
55–59	....	17	1	5	16	6	23	45	20	11	11	29	10	2	8	5	1	210
50–54	....	19	6	2	16	2	22	29	16	12	17	21	5	10	9	6	5	197
45–49	....	15	6	3	18	4	24	15	12	18	9	16	7	3	4	10	5	169
40–44	....	17	1	6	12	9	26	17	8	7	8	16	9	7	7	6	7	163
35–39	....	19	3	3	11	2	25	12	8	7	8	8	10	5	3	6	3	133
30–34	....	7	3	1	16	6	20	7	6	12	5	3	8	1	3	5	6	109
25–29	....	13	5	6	10	8	11	6	3	5	2	1	11	2	1	8	10	102
20–24	....	1	5	2	9	6	4	1	3	9	4	....	5	1	3	4	6	63
15–19	....	....	5	4	....	8	3	....	1	5	....	....	....	....	1	3	3	33
10–14	....	2	3	3	....	2	1	....	1	....	1	....	1	....	....	....	....	14
5–9	....	2	....	1	....	3	....	....	....	....	....	....	3	....	....	1	....	10
0–4	....	....	....	....	....	....	....	....	....	1	....	....	....	....	....	....	....	1
Total	....	180	71	45	197	67	282	482	214	133	183	238	82	48	89	129	51	2,491
Median	57.5	53.7	53.8	42.1	57.1	36.3	56.1	72.1	70.6	51.1	69.8	65.7	41.7	62.5	62.1	65.6	35.8	61.0
25 per cent	41.6	40.3	29.8	26.0	41.4	23.1	41.3	58.7	53.6	35.9	52.5	53.7	30.3	42.2	51.4	44.4	26.9	44.9
75 per cent	78.1	67.3	80.8	57.8	75.4	55.2	73.5	86.1	82.7	63.6	84.4	83.2	56.3	70.0	71.0	94.8	48.3	78.4

tation with improved methods and new content. It accumulates evidence of weaknesses which should be corrected in future teaching. In preparation for such a program the teachers of a department should make a study of investigations in the field carried on by others; learn to identify and formulate problems for research; acquaint themselves with the methods of scientific investigations; and learn to set up experimental conditions, construct tests, and interpret the findings. The research work should be carried on under the guidance of someone who is well qualified, usually the chairman or supervisor of the department.

It should be kept in mind that the major purpose of departmental research should be the prevention of mistakes in future teaching and the improvement of teaching on the basis of the objective findings. This should be the concern of all teachers.

Topics for research studies are not difficult to find. An investigation might grow out of the findings of a unit test which have raised questions of choice of materials or methods of instruction. Thus O'Rourke[3] began a detailed study of the unit on angles in junior high school mathematics by revising the test on this unit. He grouped the various items of the test according to the abilities to be developed in the unit and added further items until he felt that a comprehensive test of the unit had been obtained. The test in its revised form contained fifty-seven items classified into seven parts aiming to measure the abilities (1) to read angle notations; (2) to understand the fundamental space concepts of the unit; (3) to apply simple algebraic relationships to geometric figures; (4) to use the protractor with reasonable accuracy; (5) to comprehend the meaning of degree, minute, and second; (6) to use the concepts and principles of the unit in problem situations; (7) to respond to simple life-situations employing the use of some concepts of the unit.

The finished test was as follows:

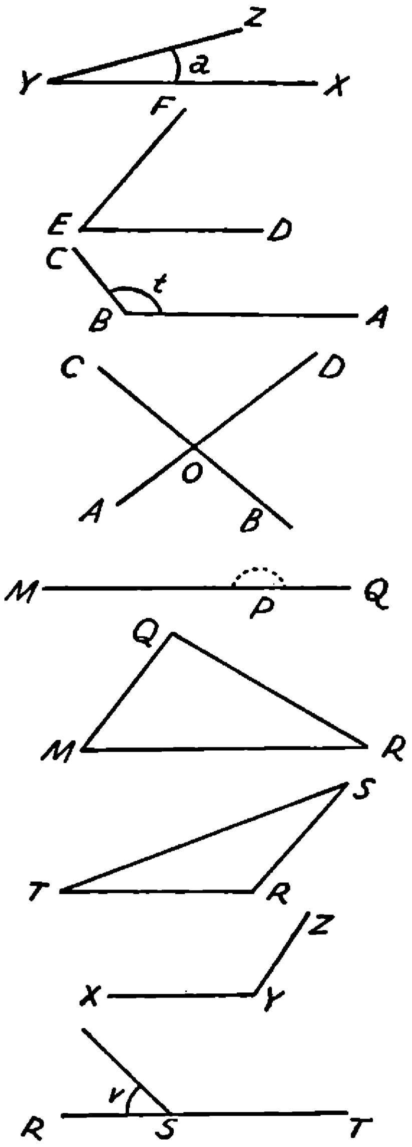
A. Reading angle notations

The items of the right column are stated in the symbols in the left column. In each blank space of the left column write the number of that item in the right column which corresponds to it. For example,

[3] Joseph M. O'Rourke, "The Unit on Angles in Junior High School Mathematics" (unpublished Master's thesis, University of Chicago, 1930).

$\angle DEF$ is found in the second figure. Hence, 2 is written in the first line.

1. $\angle DEF$ ....2............
2. $\angle AOB$ ....
3. $\angle v$
4. $\angle TRS$ ....
5. $\angle a$ ....
6. $\angle y$
7. $\angle MPQ$ ....
8. $\angle t$
9. $\angle MQR$ ....



B. Understanding of the angle concept

The following statements are incomplete. In each blank space insert the word or letter that will give a correct meaning to the statement.

1. In Figure 1.
   a) $\angle W$ is .... angle.
   b) $\angle x$ is .... angle.
   c) $\angle AOB$ is .... angle.
   d) $\angle z$ is .... angle.
   e) Line CO is .... line AB.

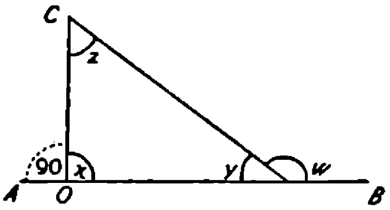


FIG. 1

2. In Figure 2.
   a) $\angle$ .... and $\angle$ .... are adjacent angles.
   b) $\angle$ .... and $\angle$ .... are opposite angles.

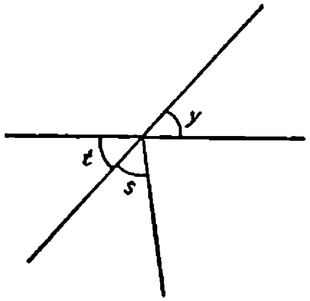


FIG. 2

3. In Figure 3, line ED is perpendicular to line EF.
   a) $\angle a + \angle b =$ ....°.
   b) Therefore $\angle a$ and $\angle b$ are called........................angles.
   c) $\angle c + \angle d =$ ....°.
   d) Therefore $\angle c$ and $\angle d$ are called........................angles.

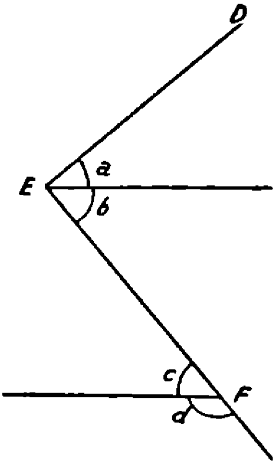
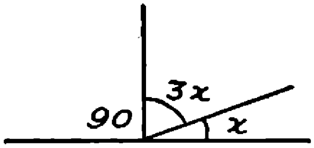


FIG. 3

C. Applying algebraic relationships to angles

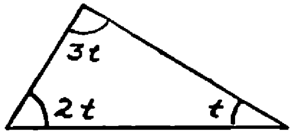
The relationships shown in the following figures can be written as equations. Write the equation that expresses the relationship in each case. Do not solve the equations.

1. Equation:..



2. Equation:..

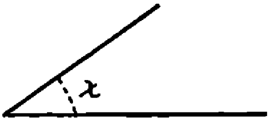
3. Equation:..



4. Write the equations for the following problems. Do not solve the equations.
   a) One of the two complementary angles is five times the other.
   Equation:..
   b) One of two supplementary angles is seven times as large as the other.
   Equation:..

D. Using the protractor

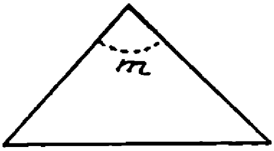
Measure the *marked* angles with your protractor. Write your answers in the blank lines.

1. 

$\angle x =$ ....°.

2.

$\angle y =$ ....°.

3. 

$\angle m =$ ....°.

Draw the following figures, using your protractor and ruler.

4. An angle of 38° at point A on line AB.

A ________________________________ B

5. A perpendicular to line MN at point P.

M ________________________________ N

P

E. Meaning of degree, minute, and second

1. Change the following to seconds:

   a) 10° = ...seconds.

   b) 12°25′30″ = ...seconds.

2. Change the following to degrees, minutes, and seconds:

   a) 24,000 seconds = ...

   b) 37,840 seconds = ...

3. Add the following:

125°25′16″
95 15 44

4. Subtract the following:

48°20′15″
20 25 20

F. Using angles in problems

In the figures below find certain facts about each figure. Your answers in each case can be discovered by using the facts written in Figure 1. Think carefully about your answers.

1. In Figure 1, $\angle f = 30°$ and $\angle e = 115°$
Therefore $\angle b =$ ....°.
and $\angle c =$ ....°.

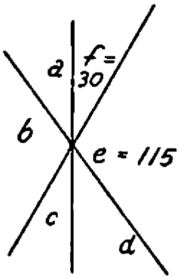


FIG. 1

2. In Figure 2, line *KL* is perpendicular to line *RS* and $\angle m = 25°$
Therefore $\angle n =$ ....°.

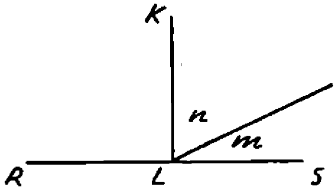


FIG. 2

3. In Figure 3, $\angle A$ is a right angle and $\angle z = 135°$
Therefore $\angle y =$ ....°.
and $\angle x =$ ....°.

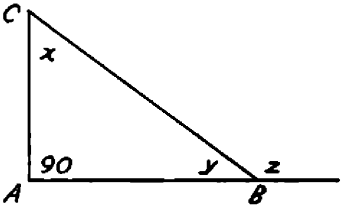


FIG. 3

4. In Figure 4, $\angle c = 45°$ and $\angle d = 45°$
Therefore $\angle b =$ ....°.
and $\angle a =$ ....°.

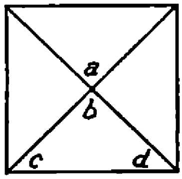


FIG. 4

5. In Figure 5, certain facts are given. From these facts it follows that
   $\angle h =$ ....°.
   $\angle k =$ ....°.
   $\angle l =$ ....°.

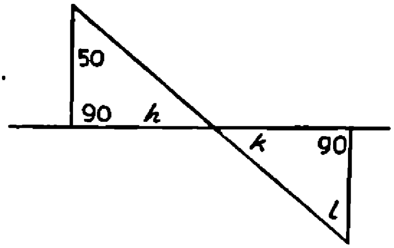


FIG. 5

G. Angles is life-situations

Below you find a number of incomplete sentences. In each blank space write the word that will make the best sentence. The missing word or words in each sentence have been taught and used in your study of angles and lines.

Sample sentence.—"Flagpoles are usually erected ——— to the ground." Here one would insert the word "perpendicular" to give the sentence proper meaning.

1. Streets usually cross each other at .... angles.
2. As the minute hand of the clock revolves it makes an .... with the hour hand that gets larger and larger.
3. The angles formed by the two lines that make the printed letter (T) are called .... angles and they are .... to each other.
4. Imagine a ladder leaning against a post. Think of the ladder and the post as forming an angle. When one would have climbed to the top of the ladder he would have reached the .... of the angle.
5. The angle formed by two consecutive spokes of an automobile wheel is an .... angle.
6. The goal posts on a football field are .... to the ground.
7. The .... angles formed by the lines of the printed letter (X) are equal.
8. When the hour hand of a clock is at six and the minute hand is at twelve, the two hands form a .... angle.
9. A trolley pole on a street car makes two angles with the wire on which it runs. One of these is an .... angle and the other is an .... angle.
10. A .... is one three-hundred-sixtieth part of a complete turn.

Frequencies of errors were computed for each pupil who had taken the test, for each part of the test, and for each class. Part E on degrees, minutes, and seconds was the hardest. Hence it was carefully studied to find ways of more effective teaching and of simplifying the materials. It was decided to omit the use of seconds in future teaching. The next part in order of difficulty was Part F on problems with angles. The others in order of decreasing difficulty were Parts G, B, D, and C. Part A on reading of angle notation was the simplest. It is possible that the procedure of developing the ability was satisfactory. However, it may be that the test needs further revision and does not give the desired information in its present form. It will be seen that the possibility of making an error is practically excluded.

Another step in the study was to make an analysis of the frequencies of errors on the specific items of the various parts. Unsatisfactory results on a part of a test are sometimes due to difficulties with but one or some of the items. The procedure of teaching the principles or processes represented by such items must therefore be carefully examined to learn when and how they may be improved.

To discover clumsy procedures and faulty mental processes used by pupils the written work was analyzed. Furthermore, pupils known to have specific difficulties were closely observed as they attacked problems involving such difficulties. The observer asked the observed pupils to explain the reason for using the procedure in question and kept a careful record of the answers. Thus evidence was obtained for developing more suitable teaching methods than had been previously employed.

The next step in a study of this type should be to test the improved methods experimentally to see if superior results are actually being attained.

The foregoing is but a rough outline of the use of tests to improve instruction. Such a study takes time and labor. However, when the supervisor and the teachers occasionally undertake the detailed study of a unit with which they did not have satisfactory results, they will not only have the satisfaction of rendering a real service to the pupils but they will personally profit a great deal by it.

Influence of college entrance examinations on instruction.—College entrance examinations have been instrumental in standardizing the traditional courses of the various mathematical subjects. Many teachers, whether they have one or several pupils preparing to take entrance examinations, will pay considerable attention to the types of problems and subject matter which the examinations contain. Some accept the entrance examinations as expressions of the opinions of experts who should be qualified to make correct decisions as to degree of difficulty and types of problems to be used in the assimilation of mathematical principles. Authors of textbooks usually include in the prefaces statements assuring the prospective users that ample provision has been made for materials which should enable the pupil to be successful in college entrance examinations.

Even a superficial study of college entrance examinations discloses a definite list of topics which are being regularly tested. Thus, the examination of the year 1931 given by the College Entrance Examination Board contains questions on factoring, fractions, formulas, linear and quadratic equations, simultaneous systems, graphs, exponents, radicals, progressions, binomial theorem, numerical trigonometry, and problem-solving. Indeed, most of these topics are represented in the examinations year after year, and since the variety of test questions is necessarily limited it is a comparatively simple matter to train pupils to make passing grades even when they are not thoroughly grounded in algebra.

If success in college entrance examinations becomes the major aim of the teacher, he may feel tempted to offer narrow courses limited to the topics represented in the examinations. This will enable a pupil to succeed in the examination although he may be poorly prepared for college work. Thus, the teacher may defeat the major purpose for which entrance examinations are written, i.e., to eliminate those who are unfit to do college work.

There is no evidence that the learning of numerous exercises illustrating a particular principle or process of algebra develops automatically complete understanding of the principle and ability to use it in new situations that will arise later in college work. Such use of entrance examinations lowers efficiency of instruction because it aims at mastery of assimilative materials and neglects the principles which the materials illustrate.

There are other reasons why the teacher should not allow his work to be influenced too much by college entrance examinations. They are necessarily conservative and do not lead in improvements. Indeed, they often delay improvement in content and instruction because they do not introduce new practices until they have been well established by the progressive teachers. For example, only in recent years have the first-year algebra examinations of the College Entrance Examination Board been simplified and brought down to the level of the average high-school Freshman. For many years some of the questions were so difficult that a pupil's chance of making a passing grade in the first-year algebra examination was small unless he had also taken the second course in algebra. Thus, in 1905 the question on complex fractions was

$$\left(\frac{\frac{a^2+ax}{2x}}{a^2-x^2}\right)\left(\frac{(a+x)^2}{4ax}-1\right).$$

In 1924 the corresponding question was

$$\frac{\frac{2ax}{x^2-a^2}+\frac{3a}{a+x}+\frac{a}{a-x}}{a-x}\times\frac{a^2-x^2}{4a}$$

There is little difference between them in difficulty and complexity. Not until 1928 was it recognized that the principles and processes involved in complex fractions may be tested by a simple exercise. In the examination of that year

$$\frac{\frac{1}{5}+\frac{1}{2a}}{4-\frac{25}{a}}$$

is given, and in 1931 the corresponding item,

$$\frac{a-\frac{a}{b}}{1-\frac{1}{b}},$$

is simpler than any of those offered in previous examinations. Long before this time the better teachers of algebra had recognized that the laws and operations with fractions could be most easily taught in connection with simple cases, and the National Committee on Mathematical Requirements recommended this step in the report of 1923. Too much attention paid to entrance examinations may delay for years certain greatly needed improvements in instruction.

Entrance examinations are instrumental in the introduction of new types of instructional materials, but they follow rather than lead in introducing the new features advocated by progressive teachers and committees. Thus, the desirability of teaching graphical representation and graphical solutions of equations in first-year algebra courses was indorsed by teachers' conferences as early as 1900, but in 1915 the College Entrance Board examination did not yet contain test items on graphs. Problems in numerical trigonometry did not appear until 1924, although during the preceding fifteen years that type of work was advocated by leaders in mathematics and discussed in books on the teaching of mathematics.

For many years courses in algebra have been criticized on account of the poor choice of verbal problems. The improvement of this phase of algebra is one of the most urgent problems to be solved, especially since ability to solve problems is one of the major objectives of algebra. There are numerous investigations which show that extremely poor results are attained in the solutions of verbal problems. Much has been written on ways of improving results. So far the contribution of entrance examinations to this problem has been small. Thus for years the College Entrance Examination Board has limited the choice of problems to a few types, such as motion problems, mixture problems, work problems, and investment problems. Furthermore, many of the problems lack the element of reality. They are mostly artificial puzzle problems of the "answer-known" type. It should be comparatively easy to prepare students to work the few types of verbal problems which occur in the examinations. The development of ability to solve verbal problems is much more difficult and involves broader experiences than offered in the problems contained in the examinations. The teacher in search of genuine problem material will receive little help from the examination questions.

The foregoing discussion may be summarized briefly as follows:

Entrance examinations in algebra contain a considerable amount of practice material which may be used profitably in the teaching and study of the subject. However, the teacher should not depend on them too much for instructional materials. The influence of the examinations as a device for improving instruction is small since they are slow to introduce simplifications and innovations recommended by committees and teachers in the field.

The same advice applies to a certain extent to the geometry examinations. It is being more and more recognized, especially since the development of intuitive geometry for the junior high school, that emphasis in demonstrative geometry should be first of all on logical thinking and training in solving original exercises. However, the College Entrance Board Examination of 1931 contains only two questions which require this type of training. Thus, a pupil might easily make a passing grade although he lacks entirely the ability to solve originals.

Two questions require the reproduction of proofs of textbook propositions. The better teachers of geometry have long ago accepted the principle that the amount of time to be spent on reproducing proof is to be reduced to leave more time for solving originals. However, the examination still seems to rank them as of equal importance.

The remainder of the examination is of a purely informational character. Pupils could do well in this part if they had studied geometry entirely by the method of intuition. The examination may thus have some influence in improving instruction, since it calls attention to the possibility of a wider use of intuitive methods to clarify meanings of geometric terms and principles.

BIBLIOGRAPHY

DIAGNOSIS AND REMEDIAL WORK

Boyce, George A. "Do Algebra Students Need Remedial Arithmetic?" *School Science and Mathematics*, XXVIII (May, 1928), 946–50.

Breed, Frederick S. "Remedial Supervision Based on a Diagnostic Survey of Instruction," *Report of the National Conference of Supervisors and Directors of Instruction: Second Yearbook* (1929), pp. 65–76.

Brownell, William A. "Remedial Cases in Arithmetic," *Peabody Journal of Education*, VII (September, 1929), 100–107.

Brueckner, L. J., and Souba, A. "A Diagnosis Sheet in Arithmetic," *Second Yearbook of the Elementary-School Principals* (1923), pp. 421–29.

Buswell, G. T. *Diagnostic Studies in Arithmetic*, "Supplementary Educational Monographs" (Department of Education, University of Chicago, 1926), pp. 188–95.

Greene, Charles E., and Buswell, G. T. *Twenty-ninth Yearbook of the National Society for the Study of Education*, chap. v. Bloomington, Ill.: Public School Publishing Co., 1930.

Greene, Harry A. "A Critique of Remedial and Drill Materials in Arithmetic," *Journal of Educational Research* (April, 1930), 262–74.

Haertter, Leonard D. "Use of the Inventory Test in Plane Geometry," *Mathematics Teacher*, XIX (March, 1926), 147–54.

Merrill, Helen A. "Why Students Fail in Mathematics," *ibid.*, XI (December, 1918), 45–46.

Minnick, J. H. "A Scale for Measuring Pupils' Ability To Demonstrate Geometrical Theorems," *School Review*, XXVII (February, 1919), 101–9.

Otto, Henry J. "Remedial Instruction in Arithmetic," *Elementary School Journal*, XXVIII (October, 1927), 124–33.

———. "Remedial Instruction in Arithmetic," *Journal of the National Education Association*, XVII (February, 1928), 87–89.

Soth, M. R. "A Study of a Pupil Retarded in Arithmetic," *Elementary School Journal*, XXIX (February, 1929), 439–42.

Spencer, P. L. "Diagnosing Cases of Failure in Algebra," *School Review*, XXXIV (May, 1926), 372–76.

Stevenson, P. R., and France, O. C. "Remedial Instruction in Arithmetic," *Educational Research Bulletin*, II (November, 1923), 291–97.

Symonds, P. M. "The Testing Program for the High School," *School Review*, XL (February, 1932), 97–108.

Welton, P. L. "A Testing Program in Elementary Algebra and Its Evaluation," *Mathematics Teacher*, XXIV (February, 1931), 69–75.

Whitcraft, L. H. "Remedial Work in High School Mathematics," *ibid.*, XXIII (January, 1930), 36–51.

Yingling, Robert W. "Diagnosis and Training in Advanced High School Algebra," *School Science and Mathematics*, XXVI (October, 1926), 729–34.

DIAGNOSIS AND RESEARCH

Coit, W. A. "Preliminary Study of Mathematical Difficulties," *School Review*, XXXVI (September, 1928), 504–9.

Crafts, Lilian L. "Causes of Failure in Plane Geometry as Related to Mental Ability," *Mathematics Teacher*, XVI (December, 1923), 481–92.

Desing, M. "Diagnostic Testing in High School Mathematics," *Educational Outlook*, IV (January, 1930), 105–30.

Dickinson, E. L., and Ruch, G. M. "An Analysis of Certain Difficulties in Factoring in Algebra," *Journal of Educational Psychology*, XVI (May, 1925), 323.

Minnick, J. H. "Certain Abilities Fundamental to the Study of Geometry," *ibid.*, IX (February, 1918), 83–90.

Rugg, H. O., and Clark, J. R. *Scientific Method in the Reconstruction of Ninth Grade Mathematics*, "Supplementary Educational Monographs" (Department of Education, University of Chicago, 1918), No. 1.

Schreiber, Edwin W. "A Study of the Factors of Success in First-Year Algebra," *Mathematics Teacher*, XVIII (February, 1925), 65–78.

Scott, Flora L. "Repetition of Errors in Algebra," *ibid.*, pp. 92–96.

Stone, Charles A. "The Construction of a Test To Measure Mathematical Ability," *School Science and Mathematics*, XXVI (November, 1926), 824–32.

Symonds, P. M. "The Psychology of Errors in Algebra," *Mathematics Teacher*, XV (February, 1922), 93–104.

Thorndike, Edward L. "The Constitution of Algebraic Abilities," *ibid.*, November, 1922, pp. 405–15.

Waples, Douglas. *Problems in Classroom Method.* New York: Macmillan Co., 1927.

Waples, Douglas, and Stone, Charles A. *The Teaching Unit.* New York: D. Appleton & Co., 1929.

Wattava, Virginia. "A Study of Errors Made in a Ninth Year Algebra Class," *Mathematics Teacher*, XX (April, 1927), 212–22.

Welte, Herbert D. *A Psychological Analysis of Plane Geometry*, "University of Iowa Monographs in Education" (State University of Iowa, Iowa City, Iowa, 1926), No. 1.

Whipple, G. M. "The Improvement of Educational Research," *School and Society*, XXVI (August 27, 1927), 249–55.

Whitcraft, L. H. "The Influence of College Entrance Examinations on the Teaching of Secondary Mathematics," *Mathematics Teacher*, XXVI (May, 1933), 257–70.

Whitney, F. L. *Methods in Educational Research.* New York: D. Appleton & Co., 1931.

Wood, B. D. *The Reliability and Difficulty of the College Entrance Examinations in Algebra and Geometry.* New York: College Entrance Examination Board, 1921.

Yocum, A. Duncan. "A First Step in Inductive Research into the Most Effective Methods of Teaching Mathematics," *School Science and Mathematics*, XIII (March, 1913), 197–200.

CHAPTER III

PROVIDING FOR INDIVIDUAL DIFFERENCES

Group instruction.—Originally instruction in American schools was individual. Each pupil prepared the lesson assigned to him and recited it to the teacher, who gave criticisms and suggestions for further study. As education grew in popularity, individual instruction was being replaced by group instruction, in which the age of the pupil was made the basis of classification. During the nineteenth century group instruction became predominant. The reason for the rapid acceptance of the plan was its promise of economy in instruction. It lowered the cost of teaching, since teachers could impart information to a group as easily as to an individual. Moreover, group discussion added life and interest to the work, and competition aroused the pupils to increased activity and effort.

Individual differences.—As advantageous as the system of mass teaching seemed to be, its defects soon became apparent. Many pupils who were able to continue in school work when given invididual instruction could not hold their own in a group and were being eliminated from school work because they failed. The major difficulty in group teaching arises largely from the wide differences among the individuals in the group. It neglects the individual, because it is planned for and addressed to the entire group.

Most noticeable to the observer are the differences in physical traits among pupils of the same group or class, especially the differences in height and weight. The truth of this observation has been repeatedly verified by careful studies and measurements.[1] In fact it is shown that for a sufficiently large number of cases the measurements of such physical traits as height, weight, chest girth, and lung capacity distribute themselves according to a law graphically represented by the probability curve. These differences are accounted for by hereditary influences and influences of environment,

[1] Bird T. Baldwin, *The Physical Growth of Children from Birth to Maturity*, "University of Iowa Studies" (1st ser.; Iowa City, Iowa: University of Iowa, 1921), Vol. I, No. 50.

as those of the home, school, social life, food, and climate. The system of heterogeneous grouping makes no allowance for them. The boy who is filled with physical energy is treated the same as the boy who is physically weak, the overgrown pupil who cannot be moved to action, and the one who is constantly embarrassed because he feels awkward and conspicuous.

Measurements of mental traits exhibit the same striking differences as those of the physical traits. They follow the same law of distribution.[2] Children of the same age disclose great intellectual differences. The range of ability of pupils of the same class, as reflected by the results of intelligence tests, is too large to be overlooked in teaching. Freeman[3] reports that the range of ability of the middle 50 per cent of ninth-grade groups varies from one to one and three-fifths times as much as the gain from one year to the next. It is not uncommon to find classes in which pupils differ by a half-dozen years in mental age.

Children differ socially.—It has been found that they pass through various stages of social reactions as illustrated in play types characteristic of different ages. The level of social age is roughly indicated by the chronological age of the pupil, but more accurately and conveniently by his physiological age. The methods of determining physiological age make use of measurements of rate of growth, dentition, height, and weight.

Other causes of individual differences.—Pupils also differ widely in previous experiences. Pretests on units of instruction disclose this fact strikingly. Thus in a unit test on angles in intuitive geometry[4] some members of a class of 18 pupils gave correct responses to 28 of the 35 parts of the test while others were acquainted with only 8 parts of the same test.

If to the foregoing differences are added the differences in attitudes, habits of study, special interests of pupils, imagination, and industry, it is easily seen why the differences in educational achieve-

[2] L. M. Terman, *The Intelligence of School Children* (New York: Houghton Mifflin Co., 1921).

[3] Frank N. Freeman, "Bases on Which Students Can Be Classified Effectively," *School Review*, XXIX (December, 1921), 735–45.

[4] E. R. Breslich, *The Technique of Teaching Secondary-School Mathematics* (Chicago: University of Chicago Press, 1931), p. 10.

ment are so great. Careful diagnoses of pupils having difficulty in school work have shown that by far the greatest obstacles to advancement are wrong attitudes toward school work and faulty habits of study. The solution of these difficulties is more an individual problem than a group problem, and no progress can be expected until the attitude of the individuals has been changed and better study habits have been acquired.

Influence of individual differences on achievement.—Measurement of achievements discloses the fact that the more heterogeneous the grouping the greater is the difference in achievement. Even when the work is done by all pupils under carefully controlled and identical conditions, the best pupils of a class usually achieve from three to five times as much as the poorest. In an arithmetical achievement test given recently to a seventh-grade group of pupils the scores of the individuals varied from fifth-grade to twelfth-grade achievement. The results of numerous studies disclose the same evidence as to difference in achievement among pupils of the same grade.

In view of the individual differences existing in heterogeneous groups, the teaching problems assume a difficult and serious aspect. They cannot be simply disposed of by planning for the average pupil with the expectation that the slow pupil will be able to keep up with him by a little extra exertion or by spending more time on his home work. The fact is that the teacher, being busy watching the group, loses sight of the individual. The chances are that the slow pupil is being rushed along at a rate too rapid to attain anything like mastery and that much of his effort will be futile. In time he will become discouraged. At the critical point when he needs assistance most the teacher is unaware of it. Thus the pupil will either drop the course or fail. Because with the ever increasing attendance in school the number of pupils at the lower end of the intelligence scale is growing, the teaching problems are becoming more serious than ever before.

Nor is justice being done to the bright pupil. He finds little incentive for exerting his powers. He has an easy time. He is being held back waiting for the slow to catch up with him. Hence, he becomes bored, impatient, and often irritated. His opportunities are being limited. Not being profitably employed, instead of de-

veloping effective habits of work, he acquires habits of idleness and becomes satisfied with low standards of attainment.

Dissatisfaction with group instruction has caused educators to look for ways of improving it, to reduce the range of achievement by paying more attention to the individuals in the group, or to group them on bases other than chronological age. During the present century various schemes have been submitted and tried, some of which will be considered in the pages which follow.

Individualized instruction.—Because of the failure of group instruction to provide for individual differences and to reach the individual, some educators have advocated the abandoning of group teaching and the return to an individual system of teaching which allows each pupil to advance at his own rate according to his own ability. Others prefer to continue group instruction but provide individual help for pupils who are unable to hold their own. The "Batavia Plan," introduced by Superintendent John Kennedy in Batavia, New York, in 1898, is of this type. It provided tutorial instruction to supplement classroom teaching.

An individual method of instruction was developed by P. W. Search in Pueblo, Colorado. He published his method in 1901.[5] All work was done in the school, and home work was discarded. Each pupil studied his lessons and recited to the teacher individually without disturbing or interrupting the others. Each had to master a fixed minimum, but there was no upper limit on the amount of work to be done. Since all pupils advanced at their own rates, it was not uncommon to find classes in which the best pupils accomplished three to four times as much as the slowest.

In 1919 Washburne began to try a system of individual instruction in Winnetka, Illinois.[6] The plan divided the curriculum into two parts, one dealing with knowledges and skills which all pupils should master, the other providing self-expression and group activities. To each part is devoted half of the school time. While pupils work in the common essentials, they proceed at their own rates.

[5] *An Ideal School* (New York: D. Appleton & Co., 1901).

[6] Carleton W. Washburne, "Burk's Individual System as Developed in Winnetka," *Adapting the Schools to Individual Differences: Twenty-fourth Yearbook of the National Society for the Study of Education* (Bloomington, Ill.: Public School Publishing Co., 1925), Part II.

Thus in a fourth-grade class a pupil may study third-grade arithmetic and fifth-grade reading. Recitations are abolished. Self-tests are provided to enable the pupil to tell when he is ready for the final unit test, to be given by the teacher. If the test is unsatisfactory, he does further study and practice work and then takes a retest. Thus the process of study and testing continues until the "goal" is mastered. The teacher's time is spent in giving individual assistance and in looking over papers.

When pupils engage in the social and creative activities, they discuss problems related to service on committees, projects, assembly programs, music, art, and physical education. No units are planned and no tests are to be taken.

The achievement of pupils taught under the Winnetka Plan was measured in a survey conducted in 1923.[7] Comparisons were made with achievement of pupils of three other comparable schools. The Winnetka pupils had certain slight advantages in some subjects, but were surpassed in others. On the whole, the individual plan had done no violence to the Winnetka pupils, but failed to produce anything like marked superiority. To the advocates of individualized instruction this has been keenly disappointing. It must be kept in mind that the large amount of interest in educational problems aroused among the teachers during the time the experiment was conducted and the discussions of such problems would give a type of training to the teachers which alone should produce considerable improvement in achievement. Winnetka's failure to show marked superiority is therefore regarded by some educators as an indication of inferiority of the individual plan.

Furthermore, the plan has proved to be expensive. The average cost per pupil in Winnetka under the individual plan was higher than that of the average pupil in one of the three school systems with which the comparisons were made. However, it is altogether possible that further improvement of the plan may be made to remove the objections to it.

Another well-known experiment with the individual plan which has aroused much interest among educators is the so-called Dalton

[7] Carleton Washburne, Mabel Vogel, and William S. Gray, *A Survey of the Winnetka Schools*, "Supplementary Educational Monographs of the *Journal of Educational Research*" (Bloomington, Ill.: Public School Publishing Co., 1926).

Plan. It was introduced in 1920 in Dalton, Massachusetts, by Helen Parkhurst.[8] Part of the plan was to enable the pupil to progress as ability permitted. There were no recitations. Another feature was the organization of the content of the curriculum in the form of contract jobs. A third feature was the laboratory method which made it possible for pupils to work together while their interests were similar. The Dalton Plan has enjoyed much popularity, since it is comparatively easy to introduce it in schools which are in favor of individual instruction. It is unfortunate that no measurements of results of the plan have been made as in the case of the Winnetka system.

Until more definite evidence of superiority of individual teaching over group teaching is available, it seems unlikely that the individual method will be adopted on a large scale. The group method has many advantages over the individual method. It is economical. The class discussions and competition in the group are stimulating. The pupils are offered much-needed and valuable social contacts and are taught how to adjust themselves until they can hold their own in a group. Grouping on the basis of chronological age comes near to social grouping. It needs to be supplemented, however, by improved methods of teaching and by a modified curriculum to permit the pupil to progress at his own rate.

Several experiments with individualized instruction in mathematics have been reported. An early account of the mathematics taught by the Winnetka Plan is given by Miss Mary M. Reese.[9] Each pupil receives a notebook in which the work he is to study has been developed. It also contains practice material. He is given an answer sheet to verify the correctness of his results. He receives help from the teacher if he asks for it. When he feels that he is ready he takes a test. If his results are unsatisfactory he returns to more practice and then takes another test. If necessary a third test is given. The teacher looks over the papers and keeps a record. In addition to the individual work done by pupils they are also engaged in social work, which occupies from one-third to one-half of the time given to mathematics.

[8] *Education on the Dalton Plan* (New York: E. P. Dutton & Co., 1922).

[9] "The Study of Mathematics under the Individual System," *Mathematics Teacher*, XV (December, 1922), 460–66.

A tryout of the Winnetka Plan has been reported by Stokes.[10] The course is divided into six main divisions, each of which is subdivided into "goals." Practice exercises for each goal and explanatory materials are then given to the pupil to study. He is also given an answer sheet. He works independently but may consult the instructor if he needs help. He continues to work sets of exercises leading to a specified goal until he does a set without error. Then he proceeds to work for the second goal, and the third, until he finishes the main division. He is then required to pass a test on all the goals of the division with a 100 per cent score. If he does not succeed he returns to more practice. This is repeated until he attains a 100 per cent mark on a final test. Then he is allowed to go on to the next main division.

No exact measurement of the results of the experiment have been reported by Stokes. However, failures were eliminated and a class of pupils who had previously failed in algebra completed the course without a single failure.

A method of individualized instruction in geometry was tried in Racine, Wisconsin, by Mary A. Potter.[11] The features of her plan are as follows. The regular geometry textbook is discarded. In place of it a loose-leaf notebook is given to the pupils in which they enter their own work, thereby making textbooks as they work through the course. Each theorem is attacked as an original exercise. After preliminary training of about eight weeks in logical reasoning, individual work on the first unit is started. Each pupil now progresses at his own rate. About one-third of the time is devoted to class recitations. The last step in the study of a unit is a review followed by the final test.

Individual work in algebra was tried in ninth-grade algebra classes in Cleveland, Ohio.[12] The method is as follows: The content of the course is divided into blocks. Mimeographed page assignments are made and distributed to the pupils. They are also

[10] C. N. Stokes, "Individual Instruction in Ninth-Year Algebra," *ibid.*, XVIII (April, 1925), 209–18.

[11] "Individualized Instruction in Geometry," *ibid.*, XIX (April, 1926), 219–26.

[12] Martha C. Cooke, "Individual Work in Algebra," *ibid.*, XXII (October, 1929), 361–64.

supplied with leading questions and answers to the problems. They are permitted to work as rapidly as they desire, checking their answers by means of the answer sheet. They may raise questions about the work. The pupil takes a test on the work of each block. If he is unsuccessful he does further study and takes a retest. When all blocks are finished he reviews and takes a test covering all blocks. Pupils who finish early study advanced work or take up recreational work of a mathematical character. The answers of the pupils to a questionnaire showed that most of them liked the individual method, that it was stimulating, and that it could be improved by modifications.

Plane geometry was taught by an individual method in Russell High School of the Southeastern State Teachers College, Durant, Oklahoma.[13] According to this method each pupil is given a work sheet with a three-week assignment. He works alone, but may secure help from the teacher or student monitors who are extra good pupils appointed by the teacher. Some phases of the various units are discussed by the class. All pupils take a test when the unit is finished. No measured results were obtained, but weaknesses of the plan were collected and recommendations for improvement were made.

In reporting the foregoing experiments with individual instruction, the various writers have occasionally called attention to the advantages and disadvantages of the plan. Briefly summarized, the advantages are:

1. Slow pupils are not driven but proceed at their own speeds.
2. Pupils are employed all the time.
3. Achievement is increased.
4. Better co-operation is established between teachers and pupils.
5. Pupils become independent workers and thinkers.

The disadvantages are:

1. Lack of group discussion and social activity.
2. Many pupils are not stimulated when working alone; others work too fast trying to keep up a record as high as that of the leaders.
3. Pupils proceed without understanding and become superficial.

[13] James H. Zant, "Individual Work in Plane Geometry," *ibid.*, XXIII (March, 1930), 155–60.

4. Pupil assistants neglect their work by spending time helping other pupils. Their explanations are often inadequate.
5. The teacher carries a heavy load trying to keep in mind the needs of so many individuals.
6. Much time of the teacher is spent on clerical tasks of keeping records.
7. The teacher is overburdened with securing the necessary lessons and exercises.

Methods of supplementing group teaching that aim to reach the individual.—Mass instruction tends to direct the teacher's attention toward the group and away from the individual pupil. For this reason many pupils fail needlessly because they did not receive assistance when they were passing through a critical period of serious difficulties. Several devices are employed to look after the individual needs of pupils in a large group. They are: opportunity classes, supervised study, diagnosis of the pupil's difficulty, and remedial treatment.

Opportunity classes are usually held after school hours.[14] A teacher of the department has charge and assumes this responsibility either as part of his regular teaching load or as a special service to be rendered to the department. The class is open to all pupils, bright or slow, who wish to come for assistance. They may stay as long as they wish, but in general should leave as soon as their difficulties have been removed. This gives the teacher time to concentrate on the few who remain. Some pupils are requested by the various teachers of the department to attend the class until certain deficiencies have been made up. These pupils receive definite assignments from their own teachers. A copy of the assignment together with the necessary comments about the individual pupil is sent to the teacher in charge of the opportunity class in order that he may give his assistance in the most effective manner.

Supervised study[15] aims to overcome the difficulties in group instruction which arise from individual differences, in particular to give the pupil the attention he needs and to make him independent by teaching him how to study. Success of supervised study depends on an effective technique which teachers must acquire to teach pu-

[14] G. L. Harris, "Supervised Study in the University of Chicago High School," *School Review*, XXVI (September, 1918), 490–510.

[15] Breslich, *op. cit.*, pp. 38–49.

pils effective habits of study, economy of time, systematic application, and habits of attention.

The method of assigning to the rapid workers in a class supplementary projects or contracts has been successfully used by some teachers. It has been observed that slow pupils do not as a rule need to work as many exercises to acquire understanding of mathematical principles as the more gifted pupils. Their slow progress, however, may keep the better pupils waiting and marking time. Hence more than one assignment is made to the class. The first is the minimum to be done by all pupils. It consists of carefully selected exercises of a simple type which lead to understanding of the principle to be acquired. A second assignment contains more difficult exercises on which the bright pupils may try their skill. A third suggests projects or contracts that may be done by those who have finished the first two assignments and have developed an interest which urges them to undertake an individual piece of work. This allows each pupil to advance at his own rate and does not interfere with group discussions on matters of importance and interest to all.

One of the causes of unsatisfactory progress of individuals in a group is that they are not really prepared for the work which they are expected to do. Thus, poor preparation in arithmetic will interfere with the pupils' progress in algebra. Lack of understanding of one unit may prove to be a serious stumblingblock in the study of the next. Inventory tests, given at the beginning of a course or a unit, provide the teacher with valuable information about the individual's former preparation. Knowledge thus acquired may be utilized in teaching and in dealing with individual cases. It shows what specific help is needed and should be given. When a considerable part of a class is poorly prepared, the class may be divided into two groups, and constant attention may be given to correcting deficiencies and to bringing the two groups to the same level as soon as possible.

Teachers find it hard to do this, because it brings to the surface all the difficulties of the class. However, the teaching situation is really simplified. For the same deficiencies exist when the class is not divided, but they do not always come to the teacher's attention. A report of successful use of the method of class division is

given by Miss Myrtle Downing.[16] She divided each of her geometry classes into three groups. Each group did the work of the lower group and an extra assignment. While one group recited the other two were studying. Pupils who finished the work of a higher group for a few days were moved to that group. A great deal of supervised study was conducted with the middle group and with the slow group, and much attention was given to teaching them effective habits of study. The method involved more planning on the part of the teacher than was usually the case with a single group, but the actual amount of labor was reduced. No measured results were reported except that the percentage of failures was low. Only 9 per cent of the pupils in her geometry classes failed as compared with 19 per cent the previous year.

Certain advantages are claimed for schemes which divide a class into groups. Transfers from one group to another for ambitious and for lazy pupils are easily made. The ambitious pupil in a low group is stimulated to try hard to advance. He sees in the good work of the better pupils models for his own. The social situation of the mixed group is not changed or disturbed by grouping.

Diagnosis and corrective treatment have as important a place in teaching as in medical work. Teachers should not proceed blindly. They should develop scientific methods of diagnosis and effective ways of remedial work. Objective tests (chap. ii) rather than mere opinion should supply the facts. The results of the tests should be carefully studied and analyzed. If necessary, a conference should be held with the pupil to discover the difficulty. The remedial treatment should be appropriate to the causes. If the pupil lacks mental ability the subject should be simplified and brought down to his level. It may be necessary to select different instructional materials. Sometimes a physical defect is at fault, and all that is necessary is to assign the pupil to a seat near the blackboard or near the front of the room. Much of the trouble pupils have is due to wrong attitudes, poor habits of study, and unwillingness to assume responsibility. The remedial treatment must in these cases find ways of bringing about the desirable changes.

[16] "Group Teaching in Geometry," *School Science and Mathematics*, XXII (May, 1922), 455–58.

Homogeneous grouping.—For purposes of simplifying instruction pupils may be grouped on bases other than age. Thus, those who have failed in a course and are repeating it require a treatment very different from that planned for pupils who take it the first time. They may be grouped together in order that instruction be adapted to their needs and abilities. Another basis for grouping takes intc consideration the special interests of pupils. Thus, separate classes may be formed for those who are preparing to go to college, to enter business, or to secure employment in industry. Owing to the fact that pupils differ widely in mental ability, it has become the practice of many schools to group them according to capacity to learn.

Grouping alone, however, cannot accomplish results unless it is accompanied by changes in subject matter and methods of teaching, adjustment of time limits, and differentiation of assignments.

Ability grouping.—With the development of tests of mental ability the way was opened for grouping pupils according to capacity to learn. Several advantages are claimed for such grouping. It is assumed that pupils of the same level of ability will work at nearly the same rate. If this is accomplished then the rapid worker need no longer be retarded. He will not develop the habit of working below the level of his ability. The slow pupil will not be rushed over the ground to keep up with the best. He will not be overburdened with work. He will understand the teacher's explanations. Failures will be reduced, if not eliminated.

Three objections are being raised against the plan of ability grouping. Many teachers consider the class situation in homogeneous ability groups unnatural and unreal. They claim that inspiration and competition so valuable in classes of pupils of varying ability is lacking. Others feel that ability grouping is unfair, especially to those placed in the low group. Many of these pupils feel humiliated. Hence they are not in the best frame of mind conducive to study. They become discouraged and indifferent, accepting failure. The third objection questions the bases on which ability groups are formed.

When a department adopts the system of homogeneous grouping, nothing should be left undone to avoid or remove the first two

objections. It is true that pupils soon find out that the slow have been separated from the bright. This fact cannot be hidden, but there is no reason why it should be pointed out to them by teachers and administrators.

Classes should not be labeled in any way, especially not as slow, medium, or bright. Factors other than mental ability should be taken into consideration in forming some of the groups. It should be generally understood that the purpose of grouping is to place every pupil where he can do his best work and where he can receive the type of instruction he needs. Methods of teaching should be chosen to fit the group. Even in the slow classes the pupils should develop a feeling of success by being assigned work which they are able to do. Encouragement rather than discouragement should be the atmosphere of every classroom. In the slow classes much attention should be paid to the development of study habits. There should be repetition of explanations, additional practice, and frequent reviews not necessary in the better groups. The slow classes should be small and they should not be turned over to new and inexperienced teachers. They should be distributed among the best teachers of the school. If they are properly conducted, it will soon appear that enough individual differences exist in each group to arouse competition and to develop leadership.

The curriculum for the various groups should be differentiated. A definitely fixed minimum should be mastered by all, and they should be given the same tests on the minimum. In addition, the stronger classes should accomplish more work, study additional topics, and engage in independent undertakings.

The plan of homogeneous grouping should include provision for readjusting pupils who have been misplaced or outgrown their respective groups. For this reason it is essential that a class of each of the groups be scheduled to come on the same period. This is not difficult to arrange in large high schools, but in the small schools it raises a serious administrative problem.

Intelligence tests as a device for determining the intelligence of high-school pupils.—If intelligence tests are to be an important factor for classifying pupils in mathematics, it is important to know the extent to which these tests provide reliable measures of intellectual capacity. A considerable number of investigations dealing with this

problem has been published. A rather extensive study was reported in 1922.[17] The tests selected for use in this investigation were the Chicago Group Intelligence Test, Form A; the Otis Group Intelligence Test, Advanced Examination, Form A; and the Terman Group Test of Mental Ability, Form A. Since they all propose to measure general intelligence, the extent to which they agree in measures of the same intelligence was indicated by the coefficient of correlation. The experiment was conducted with a group of 60 ninth-grade pupils and one of 54 seventh-grade pupils. The coefficients of correlation are shown in Table IX. It is evident that the correlations are high, varying from .69 to .85 with an average of .77.

TABLE IX

CORRELATION BETWEEN INTELLIGENCE TESTS IN THE SEVENTH AND NINTH GRADES

Tests	Seventh Grade		Ninth Grade	
	Coefficients	P.E.	Coefficients	P.E.
Chicago-Terman........	.69	.048	.77	.034
Chicago-Otis...........	.77	.038	.78	.035
Otis-Terman...........	.74	.042	.85	.024

The least agreement exists between the Chicago and Terman tests and the greatest between the Otis and Terman tests.

A series of composite intelligence scores was derived from the three intelligence tests to obtain a series which may represent more nearly the true value than any single series of scores. The correlation between Otis scores and the composite scores was found to be .92 for the ninth grade.

A detailed study was then made of the average disparity between individual scores for the same pupils in two different tests. It was found to be 6 points when measured on the Chicago scale, 11.1 points when measured on the Otis scale, and 13.9 points when measured on the Terman scale. This degree of variability led the investigators to the conclusion that considerable caution should be exercised in the use of intelligence tests for the purpose of classifying

[17] F. S. Breed and E. R. Breslich, "Intelligence Tests and the Classification of Pupils," *School Review*, XXX (January, 1922; March, 1922), 51–66, 210–26.

pupils according to intelligence. At least two good group tests should be employed. The composite of the two would provide a more reliable measure of intelligence than a single test. In the case of pupils for whom the tests show marked disagreement, additional testing will be necessary.

Intelligence tests as basis for classifying pupils.—The practical question arises as to the difference between classifications of pupils based on the foregoing tests. Table X shows the displacement of pupils for the cases of lowest, average, and highest correlation. Thus, in the middle of the table are found the displacements for the Chicago-Terman correlation for Grade IX which happens to be identical with the average correlation.

The pupils of the ninth grade were ranked from lowest to highest according to their Chicago scores. The group was then divided into three equal sections, making the number of divisions the same as the actual number of sections in first-year mathematics in the University High School. Similarly, the pupils were ranked by their Terman scores, and comparison was made to determine the number in each Chicago tertile who were in a different tertile according to the Terman scores. The number of such differences or displaced individuals is shown in the table in the column headed "Number of Displacements." It will be noticed that the percentage of pupils displaced through disparity between tests is exactly 30 for the case in which the coefficient of correlation is equal to the average. This means that of the 60 ninth-grade pupils, classified into three sections by the Chicago test, 18 were found to be in different sections from those in which they would have been if classified by the Terman test. If, as would be done in some high schools, the group of 60 pupils were here divided into two equal sections instead of three, the percentage of displacements would be reduced to 23. With the higher correlation of .85 in the case of the Terman-Otis comparison, the displacement percentage for the tertile grouping dropped to 21.7. For a division of the group into two sections of 30 each, the percentage of pupils displaced would be 20 instead of 21.7.

In the right-hand column of Table X the displacement is represented in "units," by which is meant the number of sections or tertile steps displaced. Since a pupil might be displaced two steps instead of one, it seemed expedient to present the results in terms of

these units. Displacement by more than one unit was not found to be common. There were only six individuals whose classification by one test located them two sections away from their classification

TABLE X

AMOUNT OF PUPIL DISPLACEMENT ACCOMPANYING VARIOUS DEGREES OF CORRELATION

Tertile	Number of Displacements	Units of Displacement
Grade VII. Chicago-Terman Tests $r = .69$; $N = 54$		
1	4	6
2	7	7
3	6	7
Total	17	20
Percentage	31.5	
Grade IX. Chicago-Terman Tests $r = .77$; $N = 60$		
1	4	4
2	8	8
3	6	8
Total	18	20
Percentage	30	
Grade IX. Terman-Otis Tests $r = .85$; $N = 60$		
1	4	4
2	6	6
3	3	4
Total	13	14
Percentage	21.7	

by another test. For the total number of forty-eight displacements in the three comparisons, there were found to be fifty-four units.

The results here discussed are exhibited graphically in Figures 8, 9, and 10, following the order in which the data are presented in the table. Each square in these column diagrams represents a pupil.

The classification is made in each of the three figures on the basis of the test first mentioned. An individual's score in this test appears uppermost in the square. Immediately below is his score in the

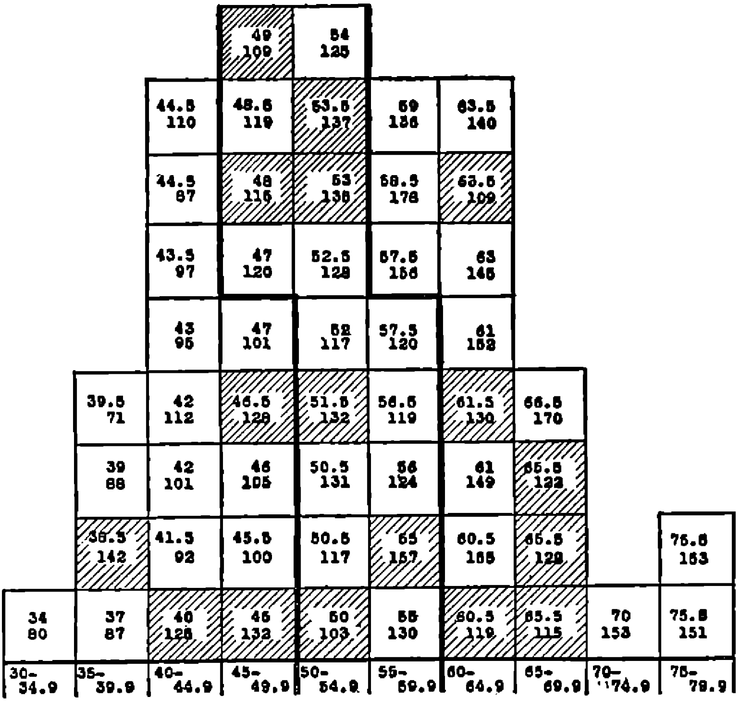


FIG. 8.—Displacements due to disparity between the Chicago and Terman intelligence tests. The correlation between the tests was .69. Fifty-four seventh-grade pupils were ranked first according to Chicago scores and then according to Terman scores. The squares represent pupils. The heavy lines mark off three equal sections of pupils classified according to Chicago scores. The upper number in each square is the Chicago score; the lower is the Terman score. The hatched squares indicate pupils displaced when the classification was based on Terman scores. There were four displaced in the first section, seven in the second, and six in the third. The total displacement was 30.5 per cent of the group.

compared test. Hatched squares indicate displaced pupils. The heavy lines mark the separation of the sections.

It may be objected at this point that these results convey a misleading impression of the reliability of the intelligence tests, since

in each case the series of scores is compared not with a series of true scores but with a series subject to error. In other words, in no case is the error of displacement due to only one of the tests. The objection is valid. In order to throw some light on this point, correlations

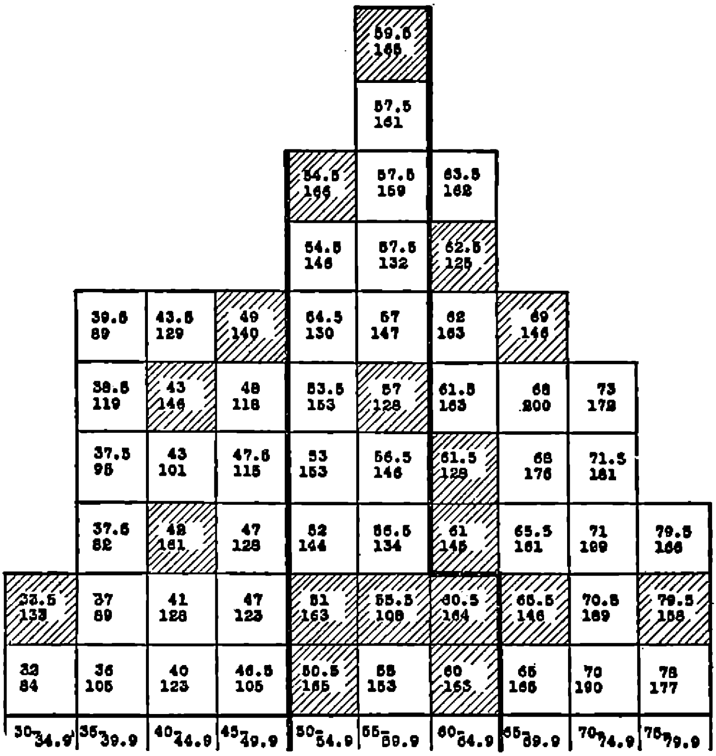


FIG. 9.—Displacements of sixty ninth-grade pupils ranked according to Chicago and Terman scores. The correlation between the tests was .77. In the three sections 30 per cent of the pupils were displaced, the number of displacements being four, eight, and six, respectively.

were computed between the test scores and the composite scores. The correlation between Otis scores in the ninth grade and the composite scores was found to be .92. This coefficient was accompanied by a pupil displacement of 13 per cent when the group was divided into three sections of 20 each, and 10 per cent when it was divided into two sections of 30 each. This indicates greater relia-

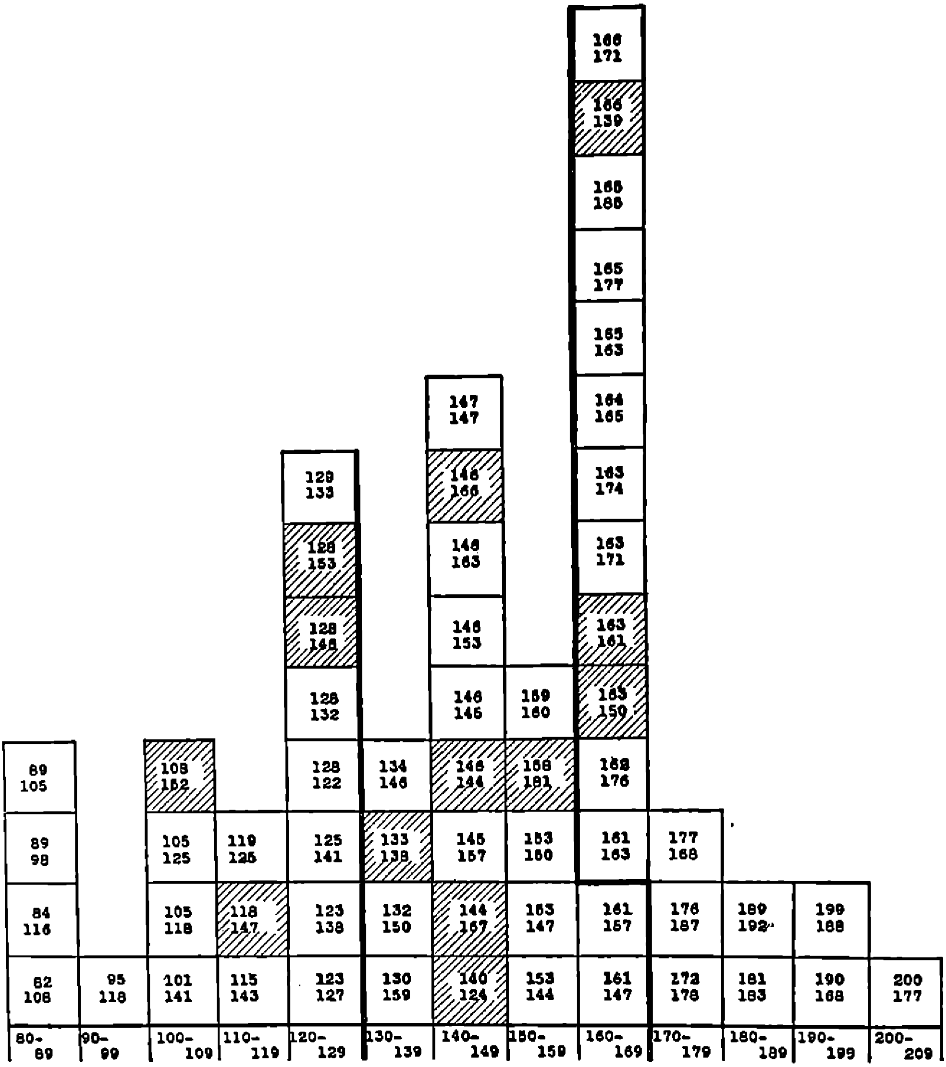


FIG. 10.—Displacements of sixty ninth-grade pupils ranked according to Terman and Otis scores. The correlation between these two tests was .85. The percentage of pupils displaced was 21.7. For the three sections the displacements were four, six, and three, respectively.

bility for the Otis Test than is apparent in Table X. The displacement for three sections is represented graphically in Figure 11.

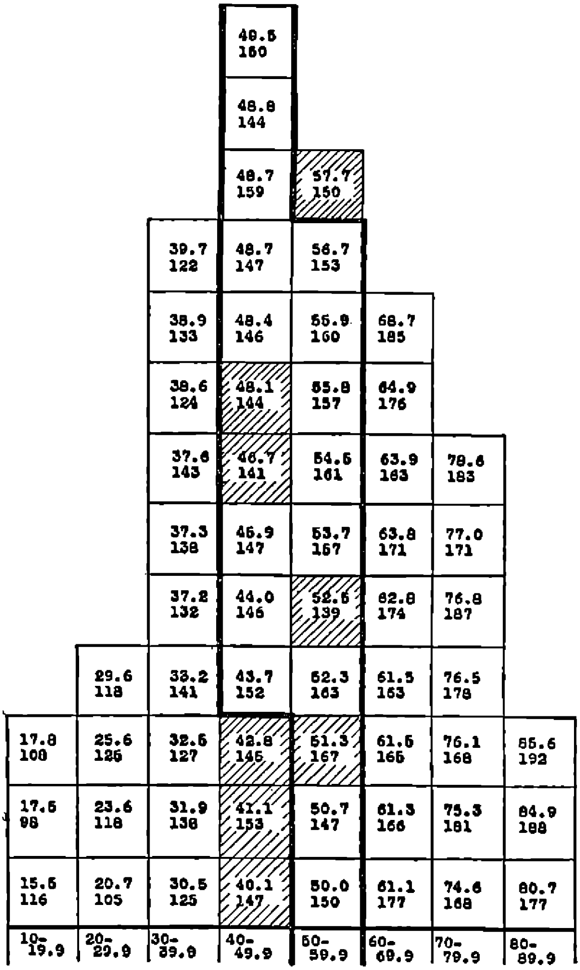


FIG. 11.—Displacements of sixty ninth-grade pupils ranked according to the composite-intelligence and Otis scores. The correlation between the two series was .92. Eight pupils, 13 per cent of the group, were displaced, the displacements for the three sections being three, four, and one, respectively.

For the purpose of obtaining a more general notion of displacement as measured with reference to the composite intelligence scores, similar data were secured for the Chicago and Terman tests. It was

found that the displacement amounted to 20 per cent for each of these tests. There was, therefore, an average displacement or error of classification amounting to about 18 per cent, owing to the variability of pupils and the inaccuracy of the measuring instrument. This means that between one-fifth and one-ninth of the pupils were not properly classified according to intelligence by the test, as judged by the criterion of composite scores. This percentage may be somewhat decreased by increasing the enrolment of sections.

Intelligence tests as a basis for predicting achievement in mathematics.—To the 54 seventh-grade pupils and the 60 ninth-grade pupils who had taken the three intelligence tests near the beginning of the semester, certain school tests were administered monthly during the semester and a final examination at the end of the semester. The tests were alike for a given phase of the work in all sections of the same grade, were administered under the direction of the author, and were scored by a key prepared by him to accompany them.

SAMPLE TEST WITH KEY FOR SCORING

I. A building 45 feet high casts a shadow 55 feet long. Find the angle of elevation of the sun by means of a scale drawing.

Key: Select the scale.......................... 1
Draw the figure.......................... 1
Find the angle.......................... 1

II. Find by similar triangles the height of a building which casts a shadow 43 feet long when a vertical 7-foot pole casts a shadow 9 feet long.

Key: Make the sketch.......................... 1
Write proportion.......................... 1
Solve the equation.......................... 1

III. Solve

$$\frac{x}{10}=\frac{19}{6}$$

Key: Multiply by 10.......................... 1
Reduce.......................... 1

Solve

$$\frac{2r}{3}+\frac{3r}{4}=23$$

Key: Multiply by 12.......................... 1
Reduce.......................... 1
Collect terms.......................... 1
Divide.......................... 1

IV. Find the angle of elevation by means of the tangent ratio when a building 118 feet high casts a shadow 143 feet long.

Key: $\tan x = \frac{118}{143}$ 1

$\tan x = .81$ 1

$x = 39$ 1

The results from the school tests were employed in two ways: First, the average score of each pupil was found in the monthly tests only. Second, the average score was found for the monthly test and the final examination, each monthly test being given a weight of one and the examination a weight of two. The correlation between the percentile composite intelligence scores and each school-test series was then computed for both grades, with the results shown in Table XI. It is observed that all the coefficients are between .30 and .40, indicating no very close relationship between

TABLE XI

CORRELATION BETWEEN SCHOOL TESTS AND PERCENTILE COMPOSITE INTELLIGENCE SCORES

Series of Scores	Seventh Grade	Ninth Grade
Monthly tests and percentile composite.........	.365	.327
School tests and percentile composite...........	.391	.315

the intelligence scores of the pupils, as represented by the percentile composite, and achievement in the school tests. If these school-test results are accepted as valid measures of achievement, the composite intelligence scores clearly do not constitute an accurate basis of classification.

The degree to which classifications of pupils based upon the two series of scores are at variance with each other is shown in Figures 12 and 13. Each pupil is indicated by a small square in the diagram. The upper number in each square is the pupil's intelligence score, the lower his school-test score. The two heavy lines running upward through the figure mark the divisions between sections. The shaded squares show the displaced pupils, that is, pupils who would be in other sections if their classification were based on their record in the school tests.

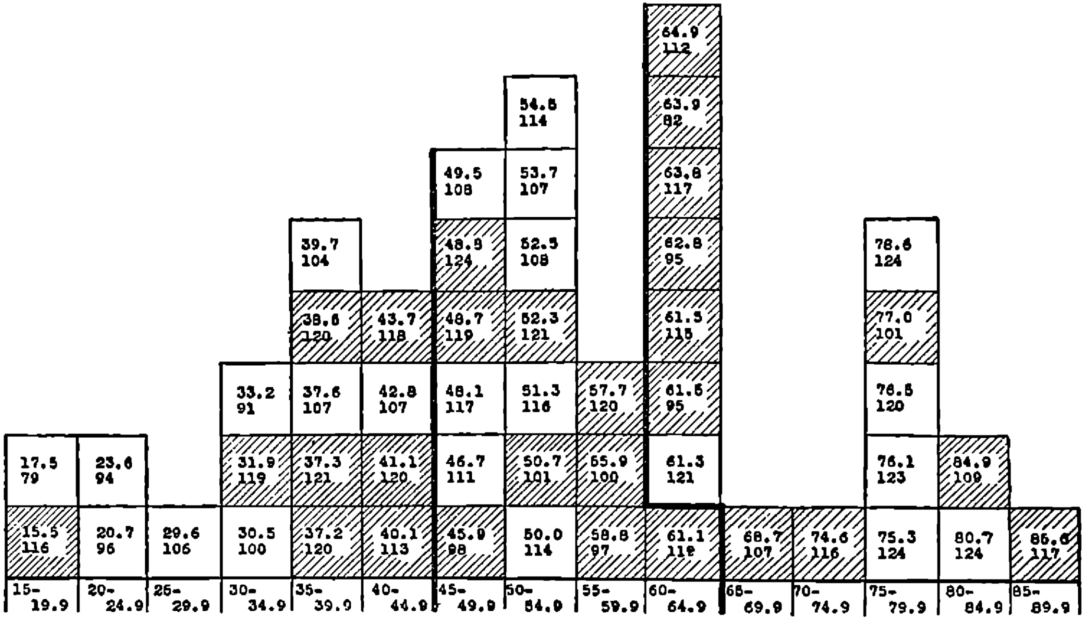


FIG. 12.—Displacements of ninth-grade pupils classified into three sections according to composite-intelligence scores. School tests were used as the criterion of educational achievement. Hatched squares represent displaced pupils. Coefficient of correlation, .31; percentage of displacement, 55.

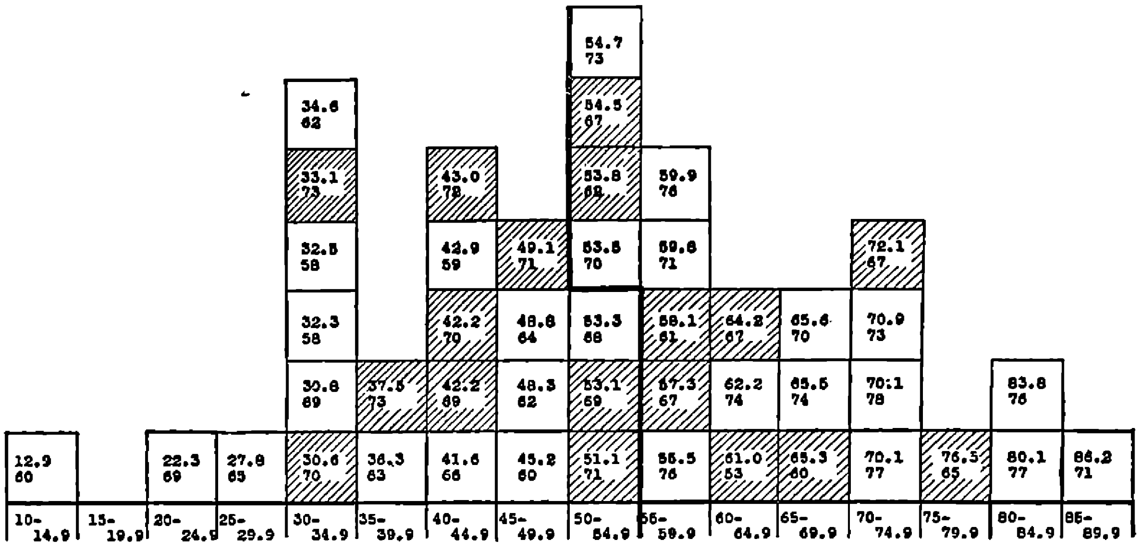


FIG. 13.—Displacements of seventh-grade pupils classified into two sections according to composite-intelligence scores. School tests were used as the criterion of educational achievement. Hatched squares represent displaced pupils. Coefficient of correlation, .39; percentage of displacement, 39.

In Figure 12, representing the ninth-grade group, the pupils are classified by the intelligence composite into three sections, corresponding to the three sections in ninth-grade mathematics. Out of the total of 51 pupils with complete records in both series of tests, 28, or 55 per cent, are found displaced. Figure 13, in a similar manner, shows the amount of pupil displacement in the seventh grade when 46 pupils are classified according to the composite intelligence scores into two sections. Eighteen, or 39 per cent, are found displaced. It should be noted that Figure 12 represents the situation where the coefficient of correlation was .31 and Figure 13 where it was .39, these being the lowest and highest coefficients respectively in Table XI.

In attempting to draw practical conclusions from the foregoing data, consideration must be given not only to the reliability of the intelligence scores, but to the reliability of the school-test scores as well. In investigations of this sort scores from conventional school tests and the ordinary marks of teachers are commonly accepted as representative of a pupil's achievement in a subject. The same sort of critical caution should be observed with reference to measures of educational achievement that is now observed in dealing with measures of intelligence.

It might be thought that careful measurements of the previous achievement of these pupils in mathematics constitute a more satisfactory basis of classification for further work in mathematics than the scores of intelligence tests. Accordingly, along with the intelligence tests given at the beginning of the semester, the Cleveland Survey Arithmetic Tests were administered to determine the computational skill of the pupils. The arithmetical-reasoning tests administered as parts of the Otis and Terman group tests of intelligence were also used to measure ability in arithmetical reasoning. In scoring the Cleveland tests the weights given by Counts[18] for the problems of the fifteen different component tests were used. The average score in the two reasoning tests was taken to represent arithmetical-reasoning ability. A score in arithmetical ability was obtained by averaging the scores in computation and reasoning

[18] George S. Counts, *Arithmetic Tests and Studies in the Psychology of Arithmetic,* "Supplementary Educational Monographs" (Chicago: University of Chicago Press, 1917), I, No. 4, 28.

with equal weight after each series had been transmuted, as previously described, into units on a percentile scale.

At the end of a semester a third check on mathematical achievement was secured. A test was given which was composed of the following examples taken from the H. B. Hotz *First-Year Algebra Scales*,[19] Series A: addition and subtraction: 1, 5, 8, 13—5 minutes; multiplication and division: 1, 2, 3, 7, 9, 11, 16—14 minutes; equation and formula: 1, 3, 4, 6, 11—6 minutes; problems: 1, 2, 4, 7—7 minutes; graphs: all—25 minutes. Examples were selected which belonged within the field covered by the pupils during the semester. The test was uniformly administered to the three sections; the papers were scored by a trained assistant, and each example in the scoring was given the value determined for it by Hotz.

Finally, inasmuch as industry seems to be one of the most important factors in scholarship, careful ratings of industry were secured from the teachers of these ninth-grade sections in mathematics. Blanks were provided with the following directions:

> Rate each of the pupils in each of the following characteristics, using the numbers 1, 2, 3, 4, and 5; 1 being the lowest rating and 5 the highest: (*a*) promptness in beginning new work, (*b*) concentration on the work once begun, (*c*) perseverance in doing assigned tasks, (*d*) accomplishment of more than the minimum requirement, (*e*) attention to questions raised and suggestions made during the class period.

Comparison of school tests and Hotz examples.—In Table XII results from the school tests and Hotz examples are compared in regard to the closeness of their relationship with other factors ordinarily considered as important for achievement in ninth-grade mathematics. The school tests in this case include both the monthly tests and the final examination. It will be seen that the Hotz examples yield a measurement more closely related to the intelligence composite than do the school tests. The respective coefficients are .56 and .31. Both tests give results fairly closely related to industry, with no appreciable difference in the closeness of the relationship. Each shows a closer relation to intelligence and industry combined than to either one of these factors alone. Industry in this computation was given a weight of four as against one

[19] "Teachers College Contributions to Education" (New York: Teachers College, Columbia University, 1918), No. 90, pp. 5 f.

for intelligence. There is no significant difference in the size of these coefficients. In relation to arithmetical ability, however, there is a marked difference between the tests. For the Hotz examples there is a coefficient of .43, and for the school tests a coefficient of .18. In the last item of the table is presented the relationship between the results from each of these tests and an average of the scores representing arithmetical ability and general intelligence. The last two series of scores were expressed in a common unit by reduction to a percentile basis before being averaged. The correlation in this case is much higher for the Hotz examples than for the school tests.

TABLE XII

SCHOOL TESTS AND HOTZ EXAMPLES IN RELATION TO VARIOUS FACTORS IN EDUCATIONAL ACHIEVEMENT

Factors	Correlation with School Tests	P.E.	Correlation with School Tests	P.E.
Intelligence composite	.31	.08	.56	.06
Industry rating	.53	....	.55	....
Intelligence composite and industry rating×4 (one section)	.61	.09	.62	.09
Arithmetical ability	.18	.09	.43	.08
Arithmetical ability and intelligence composite	.29	.09	.58	.06

In summarizing the differences between the two tests, it may be noted that the Hotz test (1) is designed to meet a situation more commonly found in typical high schools, (2) equalizes the time factor in achievement, (3) distributes the pupils more widely in the upper range of the scale, (4) yields results more closely related to arithmetical-ability scores, and (5) yields results more closely related to scores from intelligence tests. In view of these considerations the Hotz test was employed as a second criterion of educational achievement.

A comparison of various bases of classification.—The last coefficients in Table XII deserve special comment. These values have an interesting bearing on the problem of classifying ninth-grade pupils for work in mathematics. They may be considered in connection with the larger question, Which of the following, according

to the foregoing data, would constitute the most satisfactory basis of classifying these pupils: (1) intelligence tests? (2) arithmetical-ability tests? (3) intelligence and arithmetical-ability tests combined? Attention, of course, is purposely confined to bases available for use at the beginning of high-school work.

It has already been observed that the measures of arithmetical ability did not constitute a very satisfactory index of the later success of these pupils in mathematics. Further, combining the arithmetical-ability scores with the intelligence scores did not provide an appreciably more satisfactory basis of classification than the intelligence scores alone. The last two modes of classification are illustrated graphically in Figures 14 and 15. Figure 14 represents the ninth-grade pupils classified by the intelligence composite, with displacements checked against a classification according to the Hotz test. The percentage of displaced pupils was 51. When the same pupils were classified according to the combined scores in intelligence and arithmetical ability, the percentage of displaced pupils, Hotz scores being used as the criterion, was again 51. This is shown in Figure 15. These data on pupil displacement confirm the conclusion reached, namely, that a classification of the pupils based on intelligence scores would be quite as satisfactory as a classification based on a combination of intelligence and arithmetical-ability scores.

In order that the intelligence composite and the Hotz-examples series of scores may be seen more clearly in the relation to other important measures, Table XIII is provided. The correlation with industry is about the same for both, the intelligence composite correlates more highly with arithmetical ability and less highly with school tests. The coefficient of correlation with each other is .56.

While the intelligence composite was found to be as satisfactory a basis of classification at the beginning of ninth-grade mathematics as any of the other bases tested, it must be noted that most high schools would find the administering of three intelligence tests and the computation of composite scores a rather tedious process. It would be interesting to know what the loss in accuracy of classification would be, if any, in case the Otis Test were used instead of the intelligence composite. The Otis Test was therefore substitut-

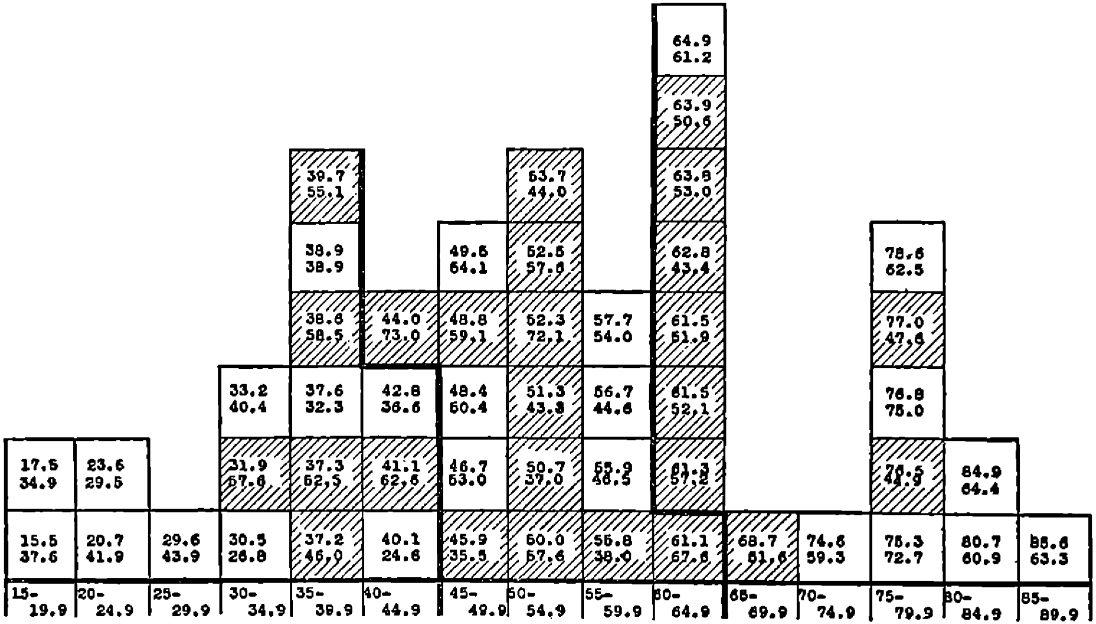


FIG. 14.—Displacements of ninth-grade pupils classified into three sections according to composite-intelligence scores. Hotz examples were used as the criterion of educational achievement. Hatched squares represent displaced pupils. Coefficient of correlation, .56; percentage of displacement, 51.

ed. The correlation between the Otis and Hotz scores was .53. The pupil displacement was 51 per cent, indicating no loss in accuracy (Fig. 16).

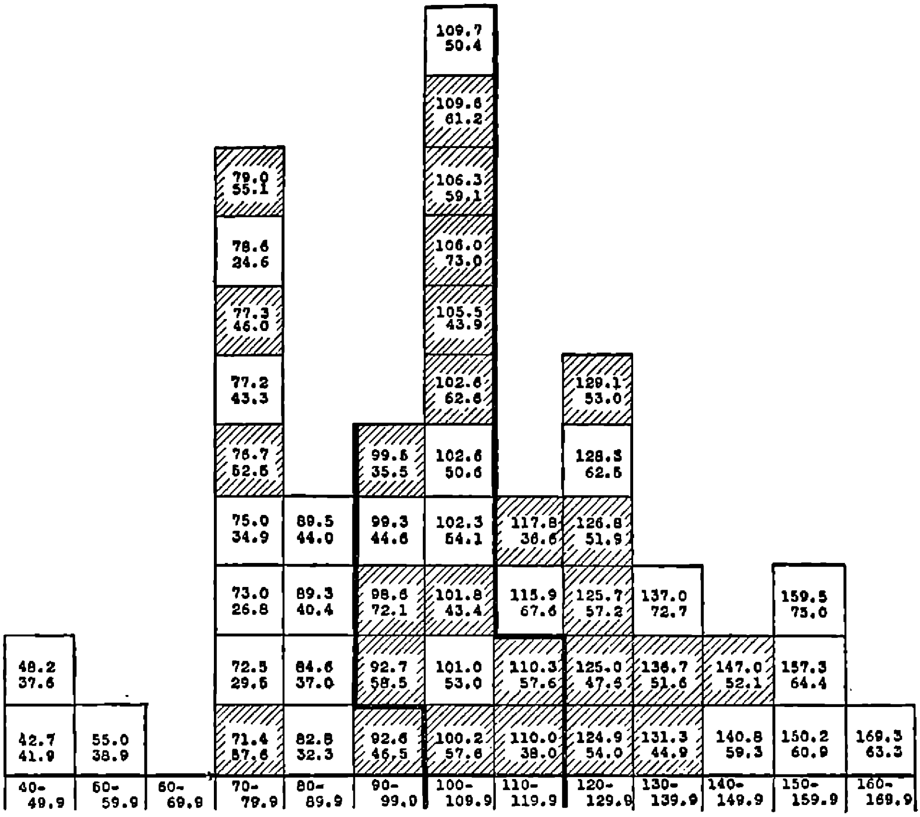


FIG. 15.—Displacements of ninth-grade pupils classified into three sections according to the combined scores in intelligence and arithmetical ability. Hotz examples were used as the criterion of educational achievement. Hatched squares represent displaced pupils. Coefficient of correlation, .58; percentage of displacement, 51.

To summarize the portion of the discussion dealing with the basis of classification, Table XIV is presented. It is seen that there is in the correlation coefficients a suggestion of improvement in the base as one proceeds from the first to the third of these comparisons.

TABLE XIII

INTELLIGENCE COMPOSITE AND HOTZ EXAMPLES IN RELATION TO OTHER FACTORS IN THE NINTH GRADE

Factor	Correlation with Intelligence Composite	P.E.	Correlation with Hotz Examples	P.E.
Arithmetical ability	.51	.058	.43	.077
Industry rating	.57	....	.55	....
School tests	.31	.081	.58	.066

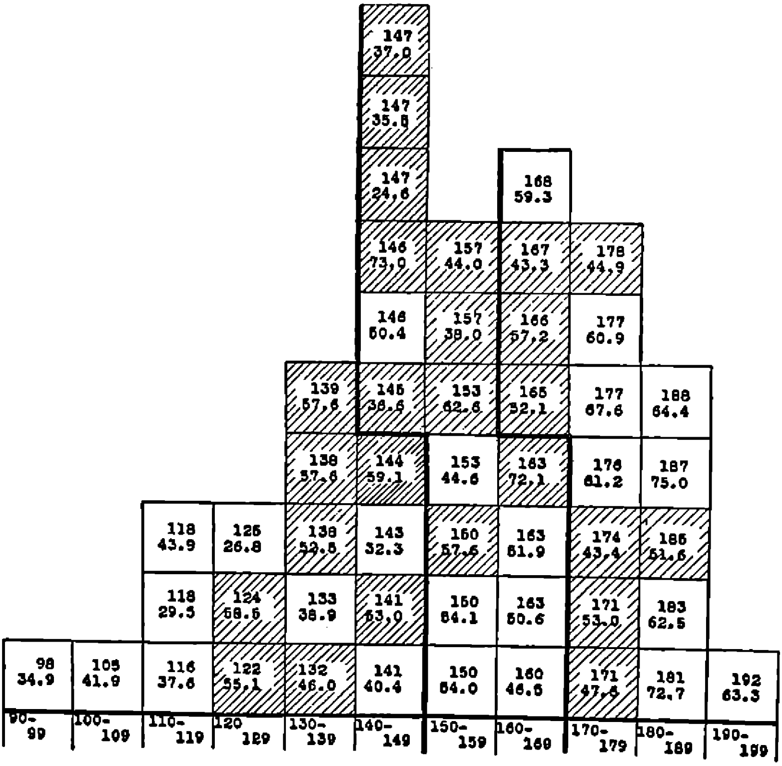


FIG. 16.—Displacements of ninth-grade pupils classified into three sections according to scores in the Otis group intelligence test. Hotz examples were used as the criterion of educational achievement. Hatched squares represent displaced pupils. Coefficient of correlation, .53; percentage of displacement, 51.

The figures on displacement, however, furnish no evidence with regard to relative superiority. The Otis Test actually classified the pupils as well as either of the other means employed.

A displacement of 51 per cent in the foregoing case seems large. One may be inclined to think that measures of intelligence should forecast scholarship with greater accuracy. They probably would if intelligence and scholarship were accurately measured. Scholarship is not a matter of intelligence alone. It is a product, as well, of such powerful emotional factors as interest and such volitional factors as perseverance. These emotional and volitional elements, only slightly if at all measured by intelligence tests, remain outside the field of intellect, to hamper or quicken the progress of the pupil in any subject.

TABLE XIV

SUMMARY OF DATA ON CORRELATION AND PUPIL DISPLACEMENT

	Coefficient of Correlation	Percentage of Displacement
Otis group and Hotz examples	.53	51
Intelligence composite and Hotz examples	.56	51
Intelligence composite and arithmetical ability; and Hotz examples	.58	51

The problem of classifying pupils by intelligence tests is obviously, then, complicated by the following conditions: (1) imperfect instruments for measuring intelligence, (2) imperfect instruments for measuring educational achievement, (3) imperfect correlation between intelligence and interest, (4) imperfect correlation between intelligence and will, (5) imperfect stability of the pupil, (6) imperfect instruction.

It seems probable, therefore, that the most accurate measurements of intelligence will not provide a reliable basis for classification under the most perfect school conditions. The Otis Test failed to classify at least 13 per cent of a group according to their intelligence. It failed to classify 51 per cent according to their educational achievement. The inaccuracy in the measurement of intelligence does not account for all of the error in the second case. If the scholarship test in the second instance be assumed to have an error as large as that of the Otis Test, and it is not conceded that it is

larger, the disparity between intelligence and scholarship is not yet explained. There is good reason from these data to believe that other factors such as those enumerated are involved in the situation and make the problem a vastly more complex one than positing a perfect relationship between two abilities such as intelligence and scholarship and measuring one of them.

All one should expect from the group tests of intelligence, so far as the general problem under discussion is concerned, is that they provide a preliminary classification, which will be subject to rectification as the scholarly ability of the pupils becomes known. This they did in the present study more economically than any other means tried, and otherwise as satisfactorily. Other things being equal, the accuracy of such classifications will probably increase considerably as the reliability of the measuring instruments, both psychological and educational, is increased.

Summary and conclusions of the foregoing study.—The study was devoted to the problem, How reliable are intelligence tests as a basis for classifying high-school Freshmen in mathematics?

Measures of intelligence were represented by the scores from the most reliable of three intelligence tests and by a composite of the three tests. The composite was assumed to provide a more reliable measure of intelligence than any single test.

A series of school tests and a test composed of examples from the Hotz algebra scales were used as the principal criteria of educational achievement.

Other data employed were industry ratings and a composite representing arithmetical ability.

The average correlation between the composite intelligence scores and the school tests was .35. Pupil displacement for the ninth grade, when divided into three classes, was 55 per cent; for the seventh grade, when divided into two classes, 39 per cent.

The correlation between the composite intelligence scores and the Hotz Test was .56. Pupil displacement for the ninth grade, when divided into three classes, was 51 per cent.

The Otis Test classified the pupils more satisfactorily than arithmetical-ability scores, and as satisfactorily as either the composite intelligence or a combination of the composite intelligence and arithmetical-ability scores.

Neither the composite intelligence scores nor the best of the intelligence tests provided a reliable basis for *permanent* classification. The error was in no case less than 50 per cent for a three-sectional classification in the ninth grade.

When the Hotz examples were used as the criterion of educational achievement, the Otis Test provided a basis as satisfactory for a *temporary* classification as any other test or combination of tests tried, and did this more economically.

Intelligence is only one af a number of important factors in educational achievement.

Besides the foregoing study numerous others have been reported in the educational literature. It seems that a reasonable procedure for grouping pupils may be about as follows:

Intelligence may be used for purposes of grouping. Then other factors should be considered.

If teachers' marks of former achievement or past school records are available they should be consulted.

A mathematical prognosis test may be administered for grouping at the beginning of a particular course.

When there is agreement between these factors the pupil may be regarded as satisfactorily placed. In cases of lack of agreement the pupil's aims and interests may be of value in making decisions. Physical condition and home environment may be of further assistance.

In all cases grouping should be regarded as tentative and provision should be made for readjustment of those whose achievement, influenced by industry, interest, and other causes, shows that they have been placed in the wrong groups.

BIBLIOGRAPHY

Arnold, H. J. "Abilities and Disabilities of College Students in Elementary Algebra," *Journal of Educational Research*, XXIII (April, 1931), 324–9.

Austin, C. M. "An Experiment in Testing and Classifying Pupils in Beginning Algebra," *Mathematics Teacher*, XVII (January, 1924), 46–56.

Barton, W. A. "The Effect of Group Activity and Individual Effort in Developing Ability To Solve Problems in First-Year Algebra," *Educational Administration and Supervision*, XII (November, 1926), 512.

Billett, Roy O. "Plans Characterized by the Unit Assignment," *School Review*, XL (November, 1932), 653–68.

Blank, Laura. "The Respective Abilities of Boys and Girls in Learning Geometry," *School Science and Mathematics*, XXXIII (February, 1933), 129–32.

Breed, Frederick S. *Classroom Organization and Management.* Yonkers, N.Y.: World Book Co., 1933.

———. "Teaching the Class and Reaching the Pupil," *School and Society*, XVIII (December, 1923), 691–96.

Buckingham, B. R. "Mathematical Ability as Related to General Intelligence," *School Science and Mathematics*, XXI (March, 1921), 205–15.

Burks, W. D. "An Experiment Comparing the Efficiency of General Mathematics with Algebra and Geometry," *Mathematics Teacher*, XVII (October, 1924), 343–7.

Burnell, E. F. "Instruction in Mathematics for Gifted Pupils," *Pedagogical Seminary*, XXIV (December, 1917), 569–83.

Callihan, T. W. "An Experiment in the Use of Intelligence Tests as a Basis for Proper Grouping and Promotions in the Eighth Grade," *Elementary School Journal*, XXI (February, 1921), 465–69.

Chastain, Loren. "Relative Values of Different Types of Assignment in a First-Year Geometry Class," *High School Principals Conference*, Vol. II (November, 1925). Bureau of Co-operative Research, Indiana University, Bloomington, Ind.

Cooke, Dennis H., and Fields, Carl L. "The Relation of Arithmetical Ability to Achievement in Algebra and Geometry," *Peabody Journal of Education*, IX (May, 1932), 355–61.

Downing, Myrtle. "Group Teaching in Geometry," *School Science and Mathematics*, XXII (May, 1922), 455–58.

Flewelling, Wilma S. "A Critical Evaluation of Individualized Instruction in Mathematics," *Mathematics Teacher*, XX (May, 1927), 5–16.

Freeman, Frank N. "Bases on Which Students Can Be Classified Effectively," *School Review*, XXIX (December, 1921), 735–45.

———. "Provision in the Elementary School for Superior Children," *Elementary School Journal*, XXI (October, 1920), 117–31.

Glass, J. M. "Classification of Pupils in Ability Groups," *School Review*, XXVIII (September, 1920), 495–508.

Grover, C. C. "Results of an Experiment in Predicting Success in First-Year Algebra in Two Oakland Junior High Schools," *Journal of Educational Psychology*, XXIII (April, 1932), 309–14.

Haertter, Leonard D. "An Experiment of the Efficiency of Large and Small Classes in Plane Geometry," *Educational Administration and Supervision*, XIV (November, 1928), 580–90.

Holzinger, K. J. "The Relative Effect of Nature and Nurture Influences on Twin Differences," *Journal of Educational Psychology*, XX (April, 1929), 241–48.

Jensen, Molton B. "The Influence of Class Size upon Pupil Accomplishment in High School Algebra," *Journal of Educational Research*, XXI (May, 1930), 337–56.

Johnson, J. T. "Adapting Instructional Material to Individual Differences in Learning," *Mathematics Teacher*, XXVI (April, 1933), 193–99.

Kerr, George P. "Results of Differentiated Curricula in Mathematics," *Proceedings of the Sixth Ohio Educational Conference*. Columbus, Ohio: Ohio State University, 1926.

Kertes, Ferdinand. "Ability Grouping in the High School," *Mathematics Teacher*, XXV (January, 1932), 5–16.

McCall, W. A., and Bixler, H. H. *How To Classify Pupils*, pp. 24–52. New York: Bureau of Publications, Teachers College, Columbia University.

Mallory, Virgil S. "A Course of Mathematics for Pupils Not Going to College," *Mathematics Teacher*, XXV (October, 1932), 340–46.

Mensenkamp, L. E. "Ability Classification in Ninth-Grade Algebra," *ibid.*, XXII (January, 1929), 38–48.

Meyers, W. C. "Curriculum Adjustments," *ibid.*, XXV (May, 1932), 264–69.

Miller, W. S., and Otto, Henry J. "Analysis of Experimental Studies in Homogeneous Grouping," *Journal of Educational Research*, XXI (February, 1930), 95–102.

Parkhurst, Helen. *Education on the Dalton Plan*. New York: E. P. Dutton & Co., 1922.

Pease, Glenn R. "Sex Differences in Algebraic Ability," *Journal of Educational Psychology*, XXI (December, 1930), 712–14.

Rogers, Agnes L. "Psychological Tests of Mathematical Ability and Educational Guidance," *Mathematics Teacher*, XVI (April, 1923), 193–205.

Schreiber, Edwin W. "A Study of the Factors of Success in First-Year Algebra," *ibid.*, XVIII (February, 1925), 65–78.

Shoesmith, Beulah I. "What Do We Owe to the Brighter Pupil?" *ibid.*, XXVI (January, 1933), 20–32.

Stokes, C. N. "Comparing the Effect of Arithmetic and General Mathematics Training in the Seventh and Eighth Grades upon Achievement in Ninth Grade General Mathematics," *School Science and Mathematics*, XXXI (October, 1931), 853–7.

Sykes, Mabel. "Differentiated Assignments," *ibid.*, XXXII (November, 1932), 863–69.

Taylor, J. F. "The Classification of Pupils in Elementary Algebra," *Journal of Educational Psychology*, IX (September, 1918), 361–80.

Washburne, Carleton W. "Burk's Individual System as Developed at Winnetka," *Adapting the Schools to Individual Differences: Twenty-fourth Yearbook of the National Society for the Study of Education*, Part II, pp. 79–80. Bloomington, Ill.: Public School Publishing Co., 1925.

Werremeyer, D. W. "Grouping Pupils According to Ability," *Mathematics Teacher*, XV (April, 1922), 237–39.

Wood, O. A. "A Failure Class in Algebra," *School Review*, XXVIII (January, 1920), 41–49.

Zant, J. H. "Individual Work in Plane Geometry," *Mathematics Teacher*, XXIII (March, 1930), 155–61.

CHAPTER IV

CHOOSING THE TEXTBOOK

The importance of textbooks in acquiring knowledge.—Knowledge is transmitted to the pupil in a variety of ways. Much of it he receives from the teacher, to whose talks and lectures he listens. The amount of information thus received and retained depends on the teacher's skill in presenting facts and on the pupil's ability to listen, to follow discussions, and to take an active part in conversation and arguments. Training in habits of listening and in participating in discussions should be offered in school, especially since in later life much information is passed on by this method. However, in the secondary school it should be only one of several methods of acquiring knowledge because it does not offer direct and genuine experiences and because talks by teachers and pupils are frequently ineffective and wasteful of class time.

Attempts to provide real experiences by letting the pupil do his own work have led to the laboratory method and supervised study. Instead of having information merely passed on to him he performs experiments. He measures, counts, compares, approximates, generalizes, and discovers mathematical facts from experiences. The method is well adapted to certain types of mathematical work and supplements other methods which do not offer direct experiences.

A third method of acquiring information in school is that of reading books. Pupils must be taught how to obtain information from books because they will use this method not only in school but long after they have finished school. They should acquire the power of mastering the printed page and its full meaning. For this reason the importance of care in the choice of the textbook cannot be overlooked. The textbook influences tremendously the material taught to the pupils. Indeed, many courses of study follow one or several carefully selected books and many teachers follow one textbook entirely to the point where the book practically replaces the teacher. If dependence on the textbook is carried too far, it usually makes instruction inadequate. The pupil needs the teacher's help to clari-

fy technical terms and difficult discussions. He should be led to read books critically and to test the conclusions reached by the authors.

A textbook cannot take the place of a good teacher but in the hands of a good teacher it becomes a powerful instrument of instruction. It will be his function to adapt the subject matter to the needs and abilities of pupils and to the conditions of the community. He will organize the content, select interesting materials, and present the tasks to be performed by the pupils in an attractive and appealing manner. He will eliminate by thoughtful planning the defects of the textbook.

The untrained teacher finds in a good textbook many suggestions as to ways of improving his teaching. Until he reaches the point where he has opinions of his own the textbook arranges the instructional materials for him, enabling him to present the facts in an ordered sequence and to give to the important phases of the course the emphasis which they deserve. Without the textbook the poorer teachers would be at loss as to the types of materials to be chosen and the best order in which they are to be arranged. The good textbook will train them in selecting and organizing materials and teach them how to teach.

To the pupil the textbook is a powerful aid. Without it he must depend on oral instruction. Often he will fail to grasp the meaning of what has been said and to understand notes taken in class. Unable to find an explanation he will be forced to memorize empty phrases which he cannot apply in situations in which understanding is required. The textbook makes it easy for him to follow the teacher as he proceeds in the course. In it he will find supplementary work arousing his interest and helping him to follow his special inclinations; references and summaries for organizing and reviewing; suggestions as to how to study the subject; and tests which he may use to estimate his progress. Even long after he has finished the course the pupil may return to his book to refresh his memory and to obtain accurate information about facts he has been unable to retain. The slow pupil and the one who for some reason is irregular in attendance find in the textbook the assistance and information needed to make up work.

From the point of view of the curriculum, good textbooks aid in

acquainting teachers with modern tendencies, new ideas, and standards of work. The growth and development of the teaching of mathematics is forcefully seen if one makes comparisons between the most recent textbooks and those published twenty or thirty years ago. The textbook has been an important factor in promoting and developing progressive ideas in mathematical education.

Public and individual production of textbooks.—Because of the importance and cost of textbooks, efforts have been made to control their production. In this country several states have published their own books and similar steps have been taken by some other countries. The arguments in favor of control are the elimination of economic waste and of unnecessary changes in subject matter. The objections are the difficulty encountered in providing for local needs when a textbook is written for use in all schools of the state, and the natural conservatism of such textbooks in introducing desirable innovations. If the system should spread, experimentation with new subject matter would be greatly discouraged. However, the probability that this should happen seems to be very small.

The tendency is to leave the production of textbooks to individuals who are experts in their respective fields. Often the author teaches the subject himself and has, therefore, the opportunity to observe personally the reactions and difficulties of the pupils using the book in manuscript form. On the basis of his experiences he may then supplement, discard, or re-write unsatisfactory subject matter. In a certain sense the pupils themselves will thus make important contributions to the textbook.

The author will endeavor to engage the co-operation and criticisms of other teachers in making revisions of the materials before they are published and when it is still a comparatively inexpensive and simple matter to incorporate the suggestions of others.

He will also take into consideration public opinion, such as the demands that pupils be thoroughly grounded in the fundamentals, that the materials conform to modern business methods and usage, and that they be of social and practical value.

He will carefully examine the recommendations of important committees, e.g., those relating to exclusion of obsolete subject matter, reduction of content to a safe minimum, and setting up of definite objectives to be attained in the various courses.

Thus, although the material is originally the author's personal selection, based on his personal interest and opinion, it will be modified considerably in the course of construction of the textbook.

Finally, the author will use his influence to induce the publishers to strengthen the mechanical features of the book. Thus he will insist that they use durable binding, clear composition, and good paper, all of which helps to make the general appearance of the book attractive.

Adopting a textbook.—The time has passed when a friendly chat with the teacher is all that is necessary to settle the question of adopting a textbook. If teachers depend entirely on the salesman, he is really the one who selects the book. To be sure, publishers and salesmen should have the opportunity to present the merits of their respective books, but no high-pressure salesmanship should be permitted. If salesmen are conscientious, they may do a real service to the schools and be a genuine help to the teachers. They will arouse the interest of the teachers and lead them to understand and appreciate the modern features and methods which their books contain. If, on the other hand, a salesman is interested merely in making a sale and if his only aim is to force his book upon the teacher, it is a rather dangerous matter to depend on his recommendation. He may actually become a retarding factor in educational progress.

The person or persons to be trusted with the selection of the books should be qualified to choose wisely. Sometimes it is the superintendent who decides the question, sometimes the principal or the supervisor. In large school systems the task is often delegated to a committee of teachers of which the supervisor is a member. The teachers who are to use the book and those who administer the work of the departments should be best qualified and should not be left out. They should be able to devise procedures for an intelligent selection of textbooks. When their decisions are made they should make joint recommendations to the final authority, e.g., the principal or the superintendent.

Various ways of selecting textbooks are described in an editorial of an issue of *School and Society*.[1] It shows that in many states a uniform list of texts is adopted either by the authority of the state

[1] "Selection of Textbooks for Public Schools," *op. cit.*, XXV (April 30, 1927), 504.

board or by a textbook commission. In a few states county adoption is found. In seventeen states local schools may select books. In California and Kansas some of the texts used are printed by the state. Four states lend texts to the districts. Nineteen states and the District of Columbia supply free texts in elementary schools and fifteen of them also to high schools. In twenty-two other states local schools may supply free texts.

Difficulties in judging textbooks.—The development of standard methods of judging textbooks is rather recent. Schemes have been proposed by various writers but it is difficult, perhaps not even practical, to develop one that would give satisfactory results in every case. A book may be suitable for use in a school whose principal aim is to prepare pupils for college, and it may not do at all for use in a technical high school. A book may satisfy the needs of an industrial community but fail to provide for the demands of a commercial or of an agricultural community. Each requires emphasis on certain types of work which may be minimized or even omitted by the others. Thus, a general scheme of ranking textbooks may be of little value in special situations. It may be misleading.

The scheme must be such as to enable teachers to compare books as to methods, content, economy of time, and numerous other features of interest in particular situations. Thus, for those groups of pupils who expect to become users of mathematics, e.g., the future engineers, scientists, and teachers of mathematics, it is important to select a text from which they may learn most effectively and with greatest economy of time. Progressive schools need to find books that conform to the most recent recommendations of committees and findings of investigators. If one school emphasizes particular phases of mathematics, e.g., arithmetic in the seventh grade, and if another school considers intuitive geometry as most suitable and desirable for that grade, different textbooks will have to be found to satisfy their needs.

There is a demand for books stressing one or more of the following: individual and class projects, unitary organization, motivation of subject matter to be taught, socialized recitations, practical exercises, problems without numbers, mental mathematics, summaries, and reviews. Methods of judging textbooks should reveal to teachers the books in which attention is paid to such features.

Schemes for analyzing the makeup of textbooks for the purpose of judging them on the basis of analysis rather than on personal preference have been drafted by individual teachers. Various school systems have devised score cards to be used in the evaluation of books. Publishers, realizing that analysis is becoming an important factor affecting the sales of books, are evaluating their own books as well as those of their competitors as to content and methods of presentation. Authors of textbooks, being aware that their products may be subjected to scorching analysis, examine all the materials in minute detail and submit them to careful experimentation before they turn the manuscript over to the publishers.

Standards for judging textbooks may be inadequate, but it is better to use them than to rely on opinion alone. Every text has some characteristics that distinguish it from others and that can be listed and described. They may be discovered by examination of the text. Some of them may be found in the author's preface and in advertising materials of the publishers. Additional data may be obtained from the salesman. This method of securing data will probably disclose the excellent characteristics better than the weak points. Book reviews may be of help. Journals of special subjects and general educational magazines bring reviews of new books which should be consulted since they may supply valuable facts.

Guiding principles for selecting textbooks.—Guiding principles have been formulated by various writers. The following illustrate the character of such principles. Herzberg[2] gives ten rules for choosing textbooks and develops them in some detail. They are:

1. To analyze subject matter as to arrangement, clearness, attractiveness, and omissions; to compare the book with the existing course of study.
2. To examine the style as to clearness, exactness, and effectiveness.
3. To analyze the book as to such pedagogical devices as practice exercises, questions, and sequence of topics.
4. To examine the mechanical makeup, e.g., binding, paper, and print.
5. To secure the opinion of teachers who have used the book.
6. To determine the qualifications of the author.
7. To compare the book with other texts.
8. To compare the book with the one in use.
9. To consider initial and later costs.
10. To arrive at decisions conservatively.

[2] M. J. Herzberg, "Ten Rules in the Choice of Textbooks," *American School Board Journal*, LIV (March, 1917), 26, 42.

Maxwell[3] presents five principles by which textbooks may be selected:

1. Textbooks must be selected with a view of meeting the ideal of those who are conducting the school.
2. A competent committee should make an exhaustive analysis of the text.
3. The members of the committee should be teachers and administrative or supervisory officers.
4. The author should be a person who has a broad knowledge of his field.
5. The text should be selected on its own merit.

Weber[4] lists eight methods which may be used in selecting textbooks and illustrates each by referring to specific studies that have been made. They are:

1. Personal analysis with reference to a few general rules or desirable qualities.
2. Determination of rank or average of a number of personal judgments.
3. Scoring on the basis of a score card.
4. Careful impersonal analysis.
5. Checking the impersonal analysis on the basis of an accepted standard of curriculum content.
6. Checking the book against the practical demands of life as shown by a survey of human activities.
7. Evaluating the book on the basis of actual class use.
8. Combination of 4 and 7.

Analysis of textbooks as basis for judgment.—Analyses of textbooks are made for various purposes. They disclose tendencies in teaching and in curriculum-making, the degree of emphasis placed on different topics, the nature of problem material, the extent to which provision is made for the needs of pupils, and many other facts of interest and importance. Several writers recommend the method of analysis for the purpose of judging the textbooks. Thus Spaulding[5] analyzes six third-grade arithmetics by dividing the content of each of the books into two classes: (1) examples of drill and textbook work in which the operations to be performed are indicated for the pupils and (2) problems in which the method of solu-

[3] C. R. Maxwell, "The Selection of Textbooks," *School and Society*, IX (January, 1919), 44–52.

[4] O. F. Weber, "Methods Used in the Analysis of Textbooks," *ibid.*, XXIV (November, 1926), 678–87.

[5] F. T. Spaulding, "An Analysis of Six Third-Grade Arithmetics," *Journal of Educational Research*, IV (December, 1921), 413–23.

tion was not directly indicated. He summarizes his findings and presents them in five interesting tables:

1. Numbers and propositions of problems and examples contained in each of the books.
2. Per cents of problems involving various operations.
3. Distribution of problems according to subject matter.
4. Comparison of the distribution of problems according to occupations with the distribution of the working population for the United States.
5. Distribution of problems involving measuring.

Jessup and Coffman[6] discuss the problem of judging arithmetic textbooks to some length and make comparisons of five textbooks. They present in tables the relative emphases given to topics, the order of topics, the number of items in each of the four fundamental processes, and the number of items in numbers, measures, fractions, and decimals. Another table shows the frequencies of arithmetical terms found in the first fifty pages of each of the books. Although the tables show more differences than agreement among the books, some conclusions are drawn showing why some of the books may be successful while the others may fail. The writers emphasize the need of the determination of scientific standards for judging books.

Score cards for judging textbooks.—An effective method of judging textbooks is to make a list of important points of good and teachable textbooks and to arrange them in a way which facilitates the ranking and scoring of specific books. It enables the teacher to tell whether one book is superior to another. An argument in favor of such a scoring device is that it gives objective data and that it may be made as comprehensive as may be desired. The difficulty lies in summarizing the ratings when a large number of points is listed and in the fact that values are necessarily relative. What may be rated as very important in one locality or situation may not be at all of importance in another.

Score cards may be designed in a variety of ways, including analyses of textbooks, listing facts stated by authors in the prefaces of their books, and consulting statements of objectives of the course for which a book is to be selected.

Mead[7] submits an outline for judging texts in reading, spelling,

[6] W. A. Jessup and L. D. Coffman, *The Supervision of Arithmetic*, chap. ix.

[7] Cyrus D. Mead, "The Best Method of Selecting Textbooks," *Educational Administration and Supervision*, IV (February, 1918), 61–69.

physiology, geography, and language, and provides a scheme for ranking the books. The nature of the outline is suggested by some of his main headings for the outline on reading. They are as follows:

I. Content
- a) Thought
- b) Form

II. Mechanical Makeup
- a) Binding
- b) Type
- c) Illustrations, etc.

Some writers present cards on which the scores are weighted. Thus Stoops[8] uses 100 as a basis and distributes the scores among three major divisions: subject matter, 50; problems, 25; and mechanical makeup, 25. Peterson[9] uses a scale of 1,000. He distributes the 1,000 units according to five major divisions: interest, 200; comprehension, 250; permanent value of subject matter, 250; value of method, 200; and mechanical elements, 100. The scores of each major division are then distributed among the subdivisions. Thus under mechanical elements he lists:

Size and clearness of print	40
Distinctness of pictures and maps	5
Size and clearness of print of marginal notes and indexes	5
Size and clearness of print of footnotes	5
Width of margins	5
Length of lines	15
Paper	15
Binding	5
Size and shape of book	5

Organizing a plan of procedure.—The problem of devising a plan of procedure for judging textbooks is complex. The first step in the solution of the problem is to select a committee of persons qualified to outline a general plan, to be followed by other committees whose function will be the evaluation of textbooks for the various mathematical courses. It is not sufficient to choose for the general committee people who have long teaching experience. They should be familiar or be able to familiarize themselves with the literature related to the problem. For example, they should know the general

[8] R. O. Stoops, "The Use of Score Cards in Judging Textbooks," *American School Board Journal*, LVI (March, 1918), 21.

[9] Allan Peterson, "A Score Card for Judging the Values of General Science Textbooks," *School Science and Mathematics*, XXII (May, 1922), 464–66.

educational objectives to which the study of the various mathematical courses should contribute, the general mathematical objectives, and the specific objectives of the courses offered by the mathematics department.[10] They should know the modern tendencies in the selection and organization of instructional materials and methods of teaching, as disclosed by objective investigations, committee reports, and recommendations of leaders in the field. They should acquire an understanding of the best methods of textbook analysis to be able to set up criteria for judging textbooks. The preparation of the committee for carrying out its functions might take several weeks. In this preparation guidance by one who has had former experience is essential. Frequent meetings will be necessary to insure progress. Practice may be obtained by a preliminary examination of one or two textbooks and by discussions of ways of analyzing them.

Aims as one of the criteria in judging a textbook.—One of the first tasks to be performed by the committee might be the formulation of the aims of the various courses. Thus, for the junior high school the mathematical aims should be stated for each of the three years. For the senior high school there should be a list of aims of demonstrative geometry, the second course in algebra, and each of the remaining courses. The lists should not be so detailed that they lose their practical value as devices for textbook analysis. It is best to make up a limited list on which all members of the committee readily agree. When the list is finished, it should be submitted to all the teachers of the department for criticisms. Upon receipt of criticisms and suggestions it should be revised and put into final form for use. Each textbook is then to be compared with the accepted aims of the course. It is unsafe to depend entirely upon the objectives stated in the prefaces of books. The materials which will contribute to the various aims should be identified by examination of the content of the textbook. In particular, it is important to note the extent to which the textbook makes the aims clear to the pupil.

The content of the textbook.—In the selection of a textbook the content is an exceedingly important factor. The recommendations

[10] Chap. v of this book; *The Technique of Teaching Mathematics in Secondary Schools*, chap. viii.

of national committees and other authorities should be carefully listed and systematized. Further suggestions will be obtained by analyzing several textbooks that have merit. The chosen topics may then be tentatively ranked according to importance. Finally, the percentage of space may be determined which is to be devoted to each topic. The list should then be checked by experts and by a local committee.

Thus, for junior high school mathematics the major topics may be: arithmetic, intuitive geometry, algebra, and trigonometry. As subtopics the following may be selected.

Arithmetic:
- The fundamental operations with whole numbers
- The fundamental operations with common and decimal fractions
- Ratio and proportion
- Square root
- Percentage and its applications to interest, commission, and discount
- Insurance, taxes, banking, and investments
- Business forms, profit and loss

Intuitive geometry:
- Lines, perimeters, circumference, and graphs
- Angles, triangles, and polygons
- Circle and its uses
- Areas
- Volumes
- Constructions with ruler and compasses

Algebra:
- Literal numbers and symbolic notation
- Signed numbers
- The operations with monomials and polynomials
- Factoring and special products
- Fractions
- Exponents and radicals
- Equations, linear in one or two unknowns
- Quadratic equations
- Formulas
- Variation

For trigonometry the topics are the trigonometric ratios and the applications to indirect measurement.

In deciding the percentages for the various subtopics the local situation should be taken into consideration. Thus, if it is intended

to give emphasis to arithmetic in the seventh grade, the space given to that subject may need to be as high as 60 per cent, or higher. If geometry is also to be emphasized, it may be assigned 40 per cent of the space and only 50 per cent may be allowed for arithmetic. If algebra is to be taught in addition to geometry and arithmetic, the percentages may be 50 for arithmetic, 35 for geometry, and 5 for algebra, the remainder being devoted to review, tests, and other purposes.

When the percentages for the major topics have been decided upon, they need to be distributed over the various subtopics and assigned to the different grades. Thus, in the seventh grade the following subtopics are usually taught and the division of time has been approximately indicated.

Arithmetic:
- The fundamental processes with whole numbers [7 per cent]
- The fundamental processes with common and decimal fractions [10 per cent]
- Ratio and proportion [3 per cent]
- Square root [2 per cent]
- Percentage and its applications to interest, commission, and discount [14 per cent]
- Insurance, banking; Taxes and investments [7 per cent]
- Profit and loss, business forms [6 per cent]

Intuitive geometry:
- Lines, perimeters, circumference, graphs [12 per cent]
- Angles and triangles [7 per cent]
- Circles and constructions [6 per cent]
- Areas [6 per cent]
- Volumes [4 per cent]

The next step in preparing a check list will be to subdivide the subtopics further. The following examples subdivided illustrate what is meant:

Percentage:
- Finding per cents of given numbers
- Finding what per cent one number is of another
- Expressing ratio as per cents and per cents as ratios
- Computing per cent of increase and of decrease
- Verbal problems using per cents

Angles:
- Angle notation
- Classifying angles
- Measuring angles
- Drawing angles of given sizes
- Constructing an angle equal to a given angle
- Constructing angles of 30°, 60°, 90°
- Bisecting angles

Graphs:
- Picture graph
- Bar graph
- Line graph
- Circular graph
- Graphs of mathematical laws

Formulas:
- Making formulas
- Translating formulas into verbal statements
- Transforming formulas
- Evaluating formulas
- Solving formulas

Trigonometry:
- Meaning of trigonometric ratios
- Reading tables of sines, cosines, tangents
- Finding inaccessible distances

The foregoing lists are applicable to junior high school mathematics. The method of making up lists for senior high school mathematics is the same. Thus, for plane geometry the following topics may be used:

- Angles and triangles
- Parallel and perpendicular lines
- Quadrilaterals
- Circle arcs, angles, and chords
- Proportional line segments
- Measurement of angles by circle arcs
- Loci and concurrent lines
- Similar figures
- Theorem of Pythagoras and its generalizations
- Areas of polygons and circles
- Regular inscribed and circumscribed polygons
- Inequalities
- Trigonometric ratios

Each of these topics should be subdivided into such divisions as theorems, axioms, exercises, applications to arithmetic, algebra and constructions, and practical applications. Lists of the minimum of required theorems and geometric constructions have been prepared by the National Committee on Mathematical Requirements and the College Entrance Examination Board. The recommendations of these two authorities are in striking agreement and are quite satisfactory for the purpose of textbook analysis. They offer a reduction of the theorems of geometry to a safe minimum. The report of the National Committee on Mathematical Requirements contains also a list of topics suitable for analysis of textbooks on intermediate algebra and trigonometry.

It is evident that the subdivision of topics may be carried out in minute detail. However, for our purposes it is not necessary to continue the process beyond the point where it ceases to be a practical device for textbook analysis. The check list should be sufficiently detailed to enable the teacher to tell the extent to which the author has incorporated the materials which correspond to modern tendencies and uses, and has eliminated that which is obsolete; to identify the material which contributes to the purposes and aims of the subject and of education; and to determine whether emphasis has been given to that which is really important and whether provision has been made for individual and local needs. The selected text should be suitable to the specific type of pupils who are to study it and to the school in which it is to be used.

Organization of materials.—It has been shown that check lists may be devised by means of which the teacher may tell how the materials of a textbook conform to the recommendations of recognized committees and experts; how they contribute to the objectives of the course; and to what extent they satisfy the real needs in the pupil's life in and out of school. It is equally important to know how thoroughly the materials have been organized for purposes of effective teaching and learning. The following points and others may be used in a check list of the organization of a course. The organization should give the course definiteness and unity. The relation between chapters following each other should be made apparent. They should not be merely a collection of topics following each other abruptly. Each chapter should be definitely related to

the whole course. Likewise the facts within each chapter should be related to the chapter as a whole. They should be grouped around broad principles to aid the pupil in understanding and retaining these principles.

The arrangement should be pedagogical. Materials should be organized from the point of view of the psychology of learning. The simple facts should come first as far as that may be possible. Abstract facts should be made concrete. Facts should be presented together when they are easily taught and learned together. For example, in algebra the work in factoring, special products, and fractions may be organized as one unit. On the other hand, care should be exercised to present but one major difficulty at a time, e.g., by separating the teaching of literal numbers from that of signed numbers. Skills once developed should be kept in use. Thus exercises, computation problems, geometric constructions, should be distributed throughout the course to provide review and further practice and to increase retention. The large and important principles of mathematics, e.g., those used in graphical representation, solution of equations and formulas, functional thinking, and problem-solving, should be kept functioning. They should not be finished in separate chapters and dropped. New concepts and definitions should be introduced when need for them arises.

The introductions to algebra and geometry should be planned with special care to make the transition from arithmetic to algebra and from intuitive to logical geometry simple and natural. When algebra is introduced in arithmetic, geometry in algebra, and trigonometry in algebra or geometry, the work should be arranged in continuous order, without interruptions. Problems should be graded according to difficulty.

Methods, discussions, and teachability.—Another factor of importance to the proper choice and arrangement of the instructional materials is the method of treatment. The presentation must stimulate interest and activity on the part of the pupils. In judging this phase of a textbook the following points will be of assistance:

The explanations of new subject matter must be so clear that they can be grasped by the pupil without assistance by the teacher. Books should be written for the pupil and should be addressed to him rather than to the teacher.

Aids to the pupil should be found throughout the book. There should be previews for the various units. The approach to new topics should not be abrupt. They should be introduced and developed gradually. The method should be largely inductive, from the simple to the complex. There should be suggestions to help the pupil in the study of difficult assignments. Illustrative examples should show the method of solution and arrangement of problems in algebra. Model proofs in geometry should suggest the form in which proofs of theorems and exercises are to be organized.

Provision should be made for individual differences. Exercises should be graded in difficulty to meet the needs of all types of pupils. Materials should be available for pupils of low, average, and high ability. There should be supplementary materials for remedial work to be done with slow pupils. The variety of the subject matter should make it possible to adapt it to the varying interests and tastes of pupils.

Appreciation and interest should be developed through the use of illustrations, historical references, and applications to industry, architecture, surveying, art, and astronomy. Pupils will thereby be stimulated to continue the study of mathematics.

Important principles of mathematics, e.g., the laws of signs and the methods of solving equations, should be carefully rationalized. When the meanings of the basic ideas have been established, they should be kept in continuous use. Definitions should be the outcomes of activities and experiences. Emphasis should be on understanding rather than on purely mechanical performance and memorization.

There should be well-directed efforts to train pupils in right methods of thinking, to establish correct habits of reading, neatness in form of arrangement and checking, and to make them intellectually independent.

Teaching devices.—To help the pupil secure mastery of fundamental facts, principles, and processes with economy of effort and time, the textbook should employ a variety of teaching devices. Illustrative examples should show how the work is to be done and arranged. The teaching of a topic should be followed by practice exercises large enough in number and variety to secure understanding. Study helps and supplementary references will contribute further to such understanding.

Tests should be used freely. There should be assimilation tests on the various elements of the units; self-tests to enable the pupil to measure his progress; and retention tests to determine whether he has made the material his permanent possession. The tests should be comprehensive enough to be diagnostic and make use of such modern devices as choice of answers and completion of incomplete statements.

Reviews and summaries should be numerous. They should be more than mere repetitions of materials in abbreviated form. They should present a new view of the content. While for teaching purposes a psychological arrangement may be used, the subject matter should be arranged in logical sequence for purposes of review.

Every good textbook should provide lists of symbols, tables of useful formulas, a table of contents, and an index so arranged that pupils may use it without difficulty.

Relation of subject matter to experiences of pupils and adults.—Textbook material should be the means of making many contacts with the pupils' experiences in and out of school and with the activities of adult life. Indeed, it may direct the pupil to look back by showing the rôle played by mathematics in the development of civilization. In the selection of the textbooks facts like the following should therefore be taken into consideration:

The study of the subject should be motivated by calling attention to the practical and social values which the pupil may derive from it. New topics may be introduced with problems which are real to the pupil so that he cultivates a desire to solve them. Much material should be offered which will develop an increasing appreciation of the value of mathematics as the pupil continues the study. From the standpoint of learning, the introduction of such material will give better results than when no incentives are offered.

The practical side of mathematics should be brought out by applications of mathematical principles in the construction of mechanical devices, and by problems of practical uses in indirect measurement, as in surveying. The problems should fall within the interest and experiences of the pupils if they are to be appreciated by them.

The immediate mathematical needs of pupils should be satisfied by problems related to his life in the home, in school, and out of

school. Many formulas and graphs should be concerned with ideas and principles which he has learned in other school subjects, especially the natural sciences.

The place of mathematics in adult life may be shown by bringing out the importance and usefulness of the subject in social life. Numerous problems should be given a social setting offering valuable facts and information. Graphs taken from business and government statistics and formulas and problems carefully selected should impress the pupil with the usefulness of mathematics in business and industry.

Illustrations.—Illustrations should aid in understanding, arouse interest, and make the subject real to the pupil. A variety of illustrations is therefore found in textbooks. Some illustrations merely stimulate interest. They give a book an attractive appearance and add to the reality of the subject. Thus, with a problem about a racing car the picture of a race track may be shown. With a problem about the dimensions of a tennis court a picture may be given in which some boys are laying out a tennis court. In general, it is considered bad practice to have too many illustrations of this type in a textbook. They do not aid the pupil in the solution of his problems. Nor are they necessary to acquire understandings.

Illustrations of subject matter with which pupils are not acquainted are very desirable. To this class of illustrations belong pictures of business forms, check blanks, and stock certificates.

Pictures of mathematicians mentioned in historical notes and of pages of ancient manuscripts have educational value and add to the pupils' interest.

In geometry illustrations may be employed to show the uses of mathematical forms and principles. Buildings containing circular or elliptic arches, bridges with supports composed of geometric figures, designs based upon geometric polygons, and machinery making use of cylinders and circles are examples of this type of illustration.

Some illustrations are expected to support the pupil's spatial imagination. For example, photographic reproductions of models, of cliffs whose heights are to be determined, and of rivers whose widths are to be found by mathematical processes may be exceedingly valuable in the solution of problems.

In demonstrative geometry it is important that the diagrams are clear-cut and accurate reproductions of the facts stated in theorems and exercises.

Vocabulary and style.—Textbooks are supposedly written for pupils and not primarily for the teacher. Hence the language and vocabulary should be suited to the pupils of the particular level of school life for whom the book is intended. Difficulties are not always mathematical. They often arise in reading and problem-solving because the pupil misunderstands or is ignorant of the meaning of the terms used or because statements of problems are vague or ambiguous. New words, especially the technical terms, should always be carefully explained before they are used in discussions and problems.

The style of presentation should be simple and clear. Explanations should be wordy and detailed. Statements should not be too abstract and compact and therefore hard to understand. Ideas should be connected with each other.

Titles and headings should be carefully worded and selected to attract interest as well as to aid understanding.

Physical aspects of the textbook.—The importance of certain physical characteristics of a textbook should not be overlooked. The book should be of a size which makes it easy to carry it about. The color of the cover should be pleasing. Light colors are objectionable because the book soon loses its pleasing appearance when it is being used.

The paper should not be glossy. Gloss is hard on the eyes and often the cause of eyestrain.

The binding should be substantial, flexible, and durable. The book should easily lie flat when it is in use. The pages should be attractive in appearance. The general type should be of a size easily read by the pupils for whom the book is intended. Different sizes of types should be used to emphasize important ideas.

Miscellaneous factors to be considered in choosing textbooks.—Although the textbook should be selected on the merits of its content, certain other information may be valuable in making decisions. There should be evidence of the authors' qualifications by experiences with pupils to write books for the level in which they are to be used. They should know the pupils, their abilities and in-

terests. Academic preparation and training are also important because they broaden the author's point of view and his ability to select and organize the materials for teaching purposes.

There is much to be said in favor of a book that has been developed in the classroom and tried out under test conditions. If possible the manuscript should be used in co-operative schools before it is published and offered to the teaching public in general.

It is not wise to be too much influenced by the publisher or by the position of the author. Not too much importance should be attached to wide use of the textbook and low cost. New books, although not widely used, are sometimes more progressive and more teachable than the old.

BIBLIOGRAPHY

SELECTION OF THE TEXTBOOKS

Editorial. "Selection of Textbooks for Public Schools," *School and Society*, XXV (April 30, 1927), 504.

Fawlkes, J. G. *Evaluating School Textbooks*. Newark: Silver, Burdett & Co., 1923.

Franzen, R. H., and Knight, P. B. *Textbook Selection*. Baltimore: Warwick & York, 1923.

Fuller, Florence D. *Scientific Evaluation of Textbooks*, pp. 1–11. Boston: Houghton Mifflin Co., 1928.

Goulet, Frank X. "Selecting the Best Textbooks," *National Education Association Journal*, XXIX (June, 1928), 183–84.

Hall-Quest, A. L. *The Textbook—How To Use and Judge It*. New York: Macmillan Co., 1918.

Herzberg, M. J. "Ten Rules in the Choice of Textbooks," *American School Board Journal*, LIV (March, 1917), 26, 42.

Ingram, E. G. "How To Judge Well Made Textbooks," *School Executives Magazine*, LI (April, 1932), 367–69.

Jensen, F. A. *Current Procedures in Selecting Textbooks*. Philadelphia: Lippincott Co., 1931.

Jessup, W. A., and Coffman, L. D. *The Supervision of Arithmetic*, chap. ix. New York: Macmillan Co., 1917.

Johnson, F. W. "A Checking List for the Selection of High School Textbooks," *Teachers College Record*, XXVII (October, 1925), 104–8.

Judd, Charles H. "Analyzing the Textbook," *Elementary School Journal*, XIX (1919), 143–54.

McLaughlin, Katherine L. "Summary of Current Tendencies in Elementary Mathematics as Shown by Recent Textbooks," *ibid.*, XVIII (June, 1918), 543–51.

Maxwell, C. R. "Choosing the Best Textbooks," *School Executives Magazine*, LI (April, 1932), 343–45.

———. *The Selection of Textbooks.* Boston: Houghton Mifflin Co., 1921.

———. "The Textbook in American Education," *Thirtieth Yearbook*, Part II. National Society for the Study of Education, VIII, 364.

———. "The Use of Score Cards in Evaluating Textbooks," *ibid.*, pp. 143–62.

Mead, C. D. "The Best Method of Selecting Textbooks," *Educational Administration and Supervision*, IV (February, 1918), 61–69.

Nutt, Hubert Wilbur. *The Supervision of Instruction.* New York: Houghton Mifflin Co., 1920.

Otis, E. M. "A Textbook Score Card," *Journal of Educational Research*, VII (February, 1923), 132–36.

Peterson, Allen. "A Score Card for Judging the Value of General Science Textbooks," *School Science and Mathematics*, XXII (May, 1922), 464–66.

Remmers, H. H., and Grant, A. "The Vocabulary Load of Certain Secondary School Mathematics Textbooks," *Journal of Educational Research*, XVIII (October, 1928), 203–10.

School of Education, New York University. "Annual Survey of Textbooks and Related Publications in Mathematics," *Journal of Educational Method*, VII (November, 1927), 30–35.

Sharwell, Truman P. "New Trends in Geometry Textbooks," *School Science and Mathematics*, XXVII (March, 1927), 261–63.

Siders, Walter R. "How Should Textbooks Be Selected?" *National Education Association*, LVIII (1920), 397–99.

Smith, David E., and Reeve, William D. *The Teaching of Junior High School Mathematics*, pp. 204–15. Boston: Ginn & Co., 1927.

Spaulding, F. T. "An Analysis of the Contents of Six Third-Grade Arithmetics," *Journal of Educational Research*, IV (December, 1921), 413–23.

Stoops, R. O. "The Use of Score Cards in Judging Textbooks," *American School Board of Journal*, LVI (March, 1918), 21.

Taber, C. W. "Selecting Textbooks," *Elementary School Journal*, XXI (June, 1921), pp. 729–30.

Weber, Oscar F. "Methods Used in the Analysis of Textbooks," *School and Society*, XXIV (November, 1926), 678–87.

Williams, Robert L. "The Selection of Mathematics Texts in the Junior High School," *School Science and Mathematics*, XXXI (March, 1931), 284–91.

CHAPTER V

BASES OF DETERMINATION OF THE AIMS AND PURPOSES OF TEACHING MATHEMATICS

The meaning of the term "objective."—Effective teaching must be based on clear and correct conception of the purposes and goals to be attained. The educational literature contains numerous discussions of the "objectives" of education and educational subjects. However, writers do not all attach the same meaning to the word. In the present volume the term designates goals to be attained and outcomes on which teaching is to be centered. Objectives should not be confused with subject matter and teaching procedures. They are the means by which the objectives are to be attained with the pupils. One of the first and most important functions of the supervisor is to set up satisfactory objectives.

The relation of objectives to teaching.—Without clearly defined aims and purposes teaching lacks purpose, instruction becomes vague and ineffective, methods are used blindly, and efforts are poorly directed. On the other hand, the character of a course will be molded by the choice of objectives. For example, many teachers of algebra stress only the technical manipulation of expressions and pay little attention to some of the other major objectives of the subject. If in a first-year algebra course the only objective is to develop facility in mechanical manipulation, the teacher will fail to attain some of the greatest values of algebra. Understanding will be sacrificed to gain time and to make room for drill. Purely mechanical rules will be substituted for the use of the fundamental processes. A premium will be placed on speed of performance. Tests will be designed to measure mainly the mechanical phases of algebra. When the teacher sets up such additional objectives as the development of abilities to do careful and accurate thinking, to draw correct inferences, to make fine discriminations, to understand functional relationships, and to use algebra in problem situations, the content and methods of the course will change accordingly. Efforts will be directed more to the acquisition and understanding

of algebraic laws and principles than to the mastery of assimilative materials. There will be tests to measure the extent to which understandings and powers have been developed.

Likewise, a course in demonstrative geometry will not be of greatest value to the pupils if the major objective of the teacher is to train them in reproducing facts, theorems, and proofs exactly as they are given in the textbook. However, if the teacher chooses as the major objective the development of power to solve original exercises and to think logically, he will plan a course of a very different character. He derives little or no comfort from the achievements of pupils who can do no more than recite glibly proofs which they do not really understand. He will spend his time teaching pupils to attack problems, to reason correctly, and to make discoveries. The tests will be tests of understanding, and of ability to apply and use geometry in practical situations.

The objectives of mathematics should be chosen wisely. To have too many objectives is as serious a mistake as to omit some that are important. It is futile to attempt more than the study may reasonably accomplish. Owing to the repeated attacks on mathematics, there has been considerable activity among the teachers to set up lists of values and aims apparently for no other reason than to answer criticisms and to make the best showing for the subject. Mathematics needs no defense. The subject is too closely related to present-day civilization to be omitted from the curriculum. Business and industry cannot get along without it. No doubt some people succeed in spite of the fact that their mathematical knowledge is limited, but those who aspire to leadership soon feel the need of it. Attempts to defend mathematics by claiming for the subject everything that is claimed for all other subjects only weaken the case of the defense.

In the discussion which follows the reader should keep in mind that the major purpose of formulating objectives is the improvement of teaching. Criticisms are not directed as much against the subject itself as against the inefficient ways of teaching it, the poor selection of instructional materials, and the faulty organization of subject matter. As long as it is not taught successfully, any subject should expect to be criticized. Criticisms should be carefully examined and steps taken to avoid them in the future, even if they

involve far-reaching changes. In this the teachers will be guided by well-defined objectives. They must keep them clearly in mind and give them constant attention in the planning of the units of the courses and of the daily work.

Methods of teaching have been discussed in detail in two volumes on the teaching of mathematics.[1] The problems of determining objectives and of selecting and organizing materials for instruction will be considered in the present volume.

Methods of determining objectives.—The determination of the objectives of mathematics offers serious difficulties. No single satisfactory method is available, and the best solution of the problem seems to be to use a variety of methods. Thus, Schorling[2] undertook to determine the objectives of junior high school mathematics on the basis of five criteria: (1) an examination of studies relating to discussions of the mathematics of Grades VII, VIII, and IX; (2) an analysis of textbooks on junior high school mathematics; (3) an analysis of courses of study; (4) the recommendations of the National Committee on Mathematical Requirements; and (5) the views of leaders in the teaching of mathematics.

Some investigators have attempted to determine objectives of mathematics by asking the opinions of prominent business men and successful professional persons. The point of view of these people is valuable and worthy of serious consideration. However, the results have been misleading, since not all successful people are sufficiently well acquainted with mathematics to know what is best, what is most valuable, and what pupils are able to understand, to assimilate, and to do. Decision on these matters must be left to experienced teachers of mathematics.

Other criteria used by investigators in determining objectives are: (1) definitions of education and of mathematics; (2) activity analyses; (3) surveys of the needs of pupils and adults; and (4) statements of specialists that may be gathered from articles in the educational journals and in the strictly mathematical magazines.

[1] E. R. Breslich, *The Technique of Teaching Secondary-School Mathematics* (Chicago: University of Chicago Press, 1930); *Problems in Teaching Secondary-School Mathematics* (Chicago: University of Chicago Press, 1931).

[2] Raleigh Schorling, *Tentative List of Objectives in the Teaching of Junior High School Mathematics with Investigations for the Determining of Their Validity* (Ann Arbor, Mich.: George Wahr, Inc., 1925).

The uses of these methods will be illustrated in the subsequent pages, but it will not be attempted to repeat long lists of objectives set up by various investigators. A summary of objectives will be found in chapter viii of *The Tcchnique of Teaching Mathematics in Secondary Schools*. Moreover, it should be remembered that a list of objectives, however complete it might be, is of no value or help to the teacher unless he plans and is able to attain them with his pupils.

Relation of mathematics to the general educational objectives.—Mathematics, as any other subject of the secondary school, must fit into the general educational scheme. It cannot stand alone, and its values will be judged largely by the contributions it makes to the attainment of the general educational objectives. The connections between the mathematical objectives and the general educational objectives must therefore be clearly recognized.

According to Spencer, the end to be achieved by education is complete living. He classifies the ultimate aims as economic efficiency, domestic efficiency, and civic-moral efficiency. He presents a list of activities by which these ends are to be acquired.[3]

Bobbitt holds that the aim of education is to prepare individuals to perform the activities which make up a well-rounded adult life.[4] He gives the following list of types of activities which are to contribute to this objective:

1. Language activities
2. Health activities
3. Citizenship activities
4. General social activities
5. Spare-time activities
6. General mental activities
7. Religious activities
8. Parental activities
9. Non-vocational practical activities
10. Vocational activities

A list of seven general objectives has been formulated by the Commission on the Reorganization of Secondary Education. They are commonly known as the "cardinal principles" of secondary edu-

[3] Herbert Spencer, *Education: Intellectual, Moral, and Physical.*

[4] Franklin Bobbitt, *How To Make a Curriculum*, p. 5.

cation.[5] The Commission takes the position that the ends to be attained are: (1) health of individual and community, (2) command of the fundamental processes, (3) worthy home membership, (4) vocational preparation, (5) citizenship, (6) worthy use of leisure time, and (7) ethical character.

Whichever of the foregoing aims of secondary education the teacher may accept, he should give them careful consideration in determining the content and methods of courses in mathematics. The problem will be to establish definite relations of mathematics to the general educational objectives and to determine the contributions which the various mathematical subjects, courses, and units may be expected to make. A method of determining pupil activities and subject matter which contribute to the objectives is illustrated in the report of the Subcommittee on Junior High School Mathematics.[6] For each major objective the Committee classifies instructional materials according to four special objectives:

A. Acquisition of fruitful knowledge
   1. Preparatory to acquiring other knowledge
   2. Knowledge functioning directly in developing dispositions and in discovering and developing abilities
   3. Knowledge which is useful in the control of situations of everyday life
B. Development of attitudes, interests, motives, ideals, and appreciations
C. Development of essential techniques, particularly with reference to judgment and problem-solving
D. Acquisition of right habits of conduct and of useful skills in living

Under each of these divisions the Committee has listed mathematical topics which relate to the major objectives of education.

In making use of the method employed by the Committee, the teacher must search his textbook for materials that contribute to the educational objectives. He must emphasize that which promises to be of superior value. He may have to supplement the textbook with materials collected from other sources.

Care must be exercised not to overcrowd courses of mathematics with materials whose value from the standpoint of general educa-

[5] U.S. Bureau of Education, *Bull. 55* (1918).

[6] Raleigh Schorling, "Report of the Subcommittee on Junior High School Mathematics," *North Central Association Quarterly*, II (March, 1928), 9–32.

tional objectives is recognized but which detract from the acquisition of the more important mathematical objectives. Commonly accepted educational objectives may become artificial and unimportant when viewed from a particular subject.[7] Thus, mathematics may have much to contribute to the vocational and social objectives of education and comparatively little to other objectives, while in the case of physical education exactly the opposite may be true. Aims which cannot be realized through the study of mathematics should either be disregarded or receive but slight attention.

General mathematical objectives.—Although mathematics, like all other subjects of the secondary-school curriculum, should make contributions to the general educational objectives, the fact should not be overlooked that the real justification for teaching the subject is the acquisition of objectives which are of a mathematical nature and which can be attained best through the study of mathematics. These objectives may be collected from the reports of important committees, from published statements made by leaders in the field, from textbooks, from articles found in the mathematical journals, and from the definitions of mathematics.

The National Committee on Mathematical Requirements[8] sets up as the "primary purpose of the teaching of mathematics" the development of powers of understanding and of analyzing quantitative and spatial relationships; habits of thought and of action; of insight into and control over our environment; of an appreciation of the progress of civilization in its various aspects. Schultze[9] stresses the development of reasoning powers, for which he considers mathematics especially fitted, and of the ability to use mathematics in science and everyday life. According to Young[10] the chief ends to be attained in mathematics are effective modes of thinking. Next in importance he ranks the practical aims, such as the ability to use mathematics in occupations, and the acquisition of valuable information.

To be sure, the usefulness of mathematics is not the only aim or

[7] *The Technique of Teaching Secondary-School Mathematics*, pp. 195–96.

[8] *Report of the National Committee on Mathematical Requirements*, pp. 6–12.

[9] Arthur Schultze, *The Teaching of Mathematics in Secondary Schools*, pp. 15–29.

[10] J. W. A. Young, *The Teaching of Mathematics*, pp. 9–52.

even the major aim of its study. Nevertheless, its importance should be recognized and the needs of practical life should be considered carefully in the selection of instructional materials. Studies that identify the mathematical needs of pupils are valuable. When these needs are known, it is possible to determine the skills, attitudes, and habits to be developed. There is much to be gained for the study of a subject if teachers include problems and applications related to life which pupils may observe and which they are able to understand. The chances that pupils will need mathematics are constantly increasing. A knowledge of mathematics is no longer required in only a few subjects, as physics and astronomy, but also in chemistry, physiology, biology, sociology, and medical research. It is the tendency of all sciences to become more and more mathematical. One reason why business men fail to make more use of mathematics is lack of training. When they have been as well trained mathematically as the scientists, they will make greater use of mathematics than at present.

As helpful as the study of the mathematical needs in practical life may be, the teacher should not overestimate its importance. They should not be used as the only basis for determining the objectives of mathematics. Values far greater than the practical are to be attained. The continued study of mathematics should bring power to the individual. The ultimate objective is power to think, to appreciate, and to do.

> The pupil who is drilled day after day in the use of number is acquiring a mode of thought which will change all of his later mental operations. The individual who has had experience with number is no longer capable of returning to the level of loose, inexact thinking that characterized his earlier methods of viewing the world. He has a mature mind. He has strengthened his mind. He is a new individual.[11]

Definitions of mathematics may be used as one of the bases for the determination of objectives. References to the following definitions are found frequently in the mathematical literature. Peirce defines mathematics as the "science which draws necessary conclusions."[12]

[11] Charles H. Judd, "The Fallacy of Treating School Subjects as Tool Subjects," *National Council of Teachers of Mathematics: Third Yearbook* (1928), pp. 4, 10.

[12] Benjamin Peirce, *American Journal of Mathematics*, IV (1881), 97.

Young says that "a mathematical science is any body of propositions arranged according to a sequence of logical deductions."[13] Sometimes mathematics has been called the science concerned with the logical deduction of consequences from the general premises of all reasoning.

It will be noted that the foregoing definitions vary, but that they have the common characteristic of stressing the logical phase of mathematics. They have nothing to say about the practical side of the subject. As an only basis for determining objectives of secondary-school mathematics they are therefore inadequate. They do, however, make it clear that the logical phase should not be slighted. A list of the major objectives of all of the mathematical subjects should contain the development of power to think, to draw correct inferences, to follow the logical train of thought of others, to form judgment independently, to acquire methods of reasoning, to understand a new situation, to sift important facts from a mass of details, to keep facts in mind, to choose discriminatingly from a number of possibilities, to analyze complex situations, and to use known or assumed facts in proving new facts. Mathematics lends itself to train these abilities because the materials are very simple to begin with and may be made to increase gradually in difficulty.

The definitions of mathematics imply that the study of mathematics helps the mind to draw correct conclusions. More opportunity for training in reasoning should be offered in mathematics than in most of the other subjects. The simple mathematical inferences are especially adapted to give such training. The definitions of mathematics demand that the pupil be taught to do his own independent thinking and not be satisfied with the conclusions others have formulated for him.

Summary.—In the foregoing pages various methods have been shown by which mathematical objectives may be determined. No single method is recommended. The best procedure seems to be to make use of all of them and to accept or reject upon careful consideration the objectives thus derived. Teachers, department heads, committees, and others who are concerned with the problem of determining mathematical objectives will save time and effort

[13] J. W. Young, *Fundamental Concepts of Algebra and Geometry* (New York: Macmillan Co.), p. 2.

by first acquainting themselves with what has been done in several extensive studies the results of which are available. The bibliography at the end of this chapter contains ample material for such a study. The reader is directed in particular to Schorling's *Tentative List of Objectives in the Teaching of Junior High School Mathematics*, the various articles by Reeve in the *Mathematics Teacher*, and the list of objectives given in chapter viii of the writer's *Technique of Teaching Secondary-School Mathematics*.

Classification of objectives.—Two methods of using objectives in instruction have been recommended as appropriate. The first makes the objectives the units of instruction. Materials are then collected and organized, and such activities are provided as enable pupils to attain the objectives. The second method begins with the organization of subject matter as it is found in textbooks and courses of study, and disturbs it only to introduce additional activities and subject matter that will contribute to certain carefully selected objectives to which the units are closely related. Thus each unit contributes definitely to the attainment of some of the general educational objectives. For example, to develop habits of neatness the teacher will refuse to accept work that is not done as neatly as the pupil is able to make it. By insisting on promptness in completing assigned tasks, in responding to requests, and in keeping appointments, the teacher trains the pupil to be prompt. To secure accuracy the teacher will not overlook inaccuracy in computation, reading, solving problems, and statements made by pupils in written and oral recitations. Appreciation is developed from numerous illustrations showing that mathematics is essential in business, in vocations, and in various school subjects, and that it has played an important part in the work of civilization. By training pupils to work with groups, to understand and respect the rights of others, to assume social obligations, to have regard for the will of the majority, to co-operate, to practice self-control, and to assume leadership, the teacher will make contributions to the social objective. He is also in a position to contribute to the vocational objectives by aiding pupils in acquiring types of knowledge and skills needed in life's work and by teaching them to follow directions, to take pride in good work, to develop the desire to improve and excel, and to concentrate on the tasks before them. To secure the fullest co-

operation the pupil should be taken into confidence in regard to the objectives to be attained.

What has been said of general educational objectives applies also to subject objectives, course objectives, and unit objectives. The teacher will go about his work intelligently if he familiarizes himself with a list of objectives and consults it freely in planning the course and each unit of the course. For example, without a thorough knowledge of objectives the teacher of an eighth-grade class, in planning a unit on areas and volumes of the common solids, may be satisfied if the pupils understand certain formulas and are able to apply them in practical problems. On the other hand, if he is familiar with the objectives of the course he will find in this unit many opportunities to contribute to them. Thus, if he is aware that understanding of the formula is a major mathematical objective, he will give special emphasis in the unit to evaluating, solving, and deriving formulas. Squares and cubes will be taught so as to lead the pupil on toward mastery of the exponential notation. Training will be given in methods of solving verbal problems. Functional relationships will be stressed. Likewise, the other units will present situations which should be directly related to some of the major mathematical objectives.

Various schemes of classifying objectives are used. The traditional way is to group them according to disciplinary, utilitarian or practical, and cultural aims of mathematics. The subcommittee on junior high school mathematics has classified them as social, vocational, leisure-time, and health objectives. Schorling listed attitudes, concepts, abilities, and information as mathematical objectives.

Objectives may be classified conveniently as powers to be developed, appreciations, understandings, and attitudes. Each of these four major classes may then be further subdivided. Thus, under "powers" may be listed powers to be developed through the study of mathematics in general, and through the study of the various mathematical subjects, such as arithmetic, algebra, geometry, and trigonometry. The following partial outline[14] illustrates what is meant.

[14] See *The Technique of Teaching Secondary-School Mathematics*, chap. viii.

I. Powers to think and to do
   1. Power to be developed by all secondary subjects
   2. Power to be developed by the mathematical subjects in general
   3. Specific mathematical power, as
      a) Arithmetical
      b) Algebraic
      c) Geometric
      d) Trigonometric

II. Appreciations to be developed
   1. By all mathematical subjects
   2. Specifically in arithmetic
      In algebra
      In geometry
      In trigonometry and higher subjects

III. Understandings to be developed
   1. In arithmetic
   2. In algebra
   3. In geometry
   4. In trigonometry and higher subjects

IV. Attitudes

V. Habits and ideals

VI. Skills

The foregoing list may be made as complete as the teacher finds it useful.

Measuring the acquisition of objectives.—Until teachers are able to measure the extent to which objectives are actually attained by pupils, teaching will be ineffective. Numerous excellent tests are available to measure results in the manipulative and informational aspects of mathematics. The explanation for this is partly that they are easily tested. It is a simple matter to construct tests to determine the extent to which pupils can perform such processes as resolving into factors well-known algebraic binomials and trinomials, finding the roots of linear and quadratic equations, solving certain types of verbal problems, and recalling from memory formulas and theorems. The ease with which this phase of mathematics may be tested and measured is perhaps the reason why so much stress has been placed on it in teaching. However, as important as these processes may be, they are not the only objectives of mathematics. They are the means for developing understandings, appreciations, and powers of far-reaching nature.

When leaders in mathematics state the objectives of the teaching of mathematics, they invariably speak of culture, discipline, power of reasoning, understandings, control over environment, modes of thinking, appreciations, and ability to apply, as the chief ends to

be attained. However, tests to measure attainment of such objectives are lacking. We are just beginning to learn how to construct them. Thus, little has been done to measure the cultural values claimed for mathematics. The problem may be difficult but its solution should not be impossible. Indeed, it is being worked out in some of the secondary-school subjects. For example, tests have been constructed to measure such vague objectives as appreciation of literature and social attitudes.[15] They are of real assistance to the teacher because they suggest specific methods and concrete materials which he may use to develop appreciations and attitudes in his courses. The outcome is therefore not left entirely to chance. As soon as suitable tests are available, teachers will become more intelligent about ways of attaining the objectives.

If the teacher wishes to attain the cultural values of mathematics with his pupils, he might begin with an analysis of definitions of culture to identify definite aspects and characteristics of the concept which may be taught and tested in mathematics. It may not be possible to find a definition that is entirely satisfactory and the teacher may have to be satisfied with statements that merely approach a real definition. A search of the literature for articles dealing with the subject and of books on the teaching of mathematics will disclose many helpful suggestions. Thus, Mathew Arnold says: "Culture is knowing the best that has been thought and said." In discussing the concept one writer expresses the view that "everything is of cultural value which makes life fuller and richer; which places us in a position of greater harmony with our surroundings; and which leads us to a better understanding of nature, of our own individual development and of the development of the race."[16] Other definitions may be added to these two. The questions confronting the teacher are: How may the subject of mathematics contribute to the various phases of culture, and how may they be developed through the study of mathematics? A pupil may recite day after day the proofs of propositions in geometry without ever

[15] H. C. Hill, *Tests in Civic Information and Attitudes* (Bloomington, Ill.: Public School Publishing Co.,); Hannah Lagosa and Martha McCoy, *Seven Tests for Appreciation of Literature* (University High School, University of Chicago).

[16] A. J. Kempner, "The Cultural Value of Mathematics," *Mathematics Teacher*, XXII (March, 1929), 127–45.

enjoying the satisfaction which comes from the knowledge that the propositions form a logical sequence and that by learning to use the laws of logic he is acquiring the ability to draw conclusions without depending on the judgment of his teacher and classmates. Unless the teacher develops in his pupils an appreciation of the logical system and of the power which comes to those who learn to think logically and independently, they may never see the beauty of geometry, and certain cultural values of geometry will be lost to them.

Likewise the pupil should be led to regard the subject of algebra as more than a tool with which he works problems, and whose major purpose is to develop skill in manipulating algebraic symbols.

It is true that much is taught in mathematics which is of no practical value to many pupils. However, no educated person should remain ignorant of the close connection between the development of mathematics and the progress of the race, the struggle of the human race to improve symbols, the power stored up in the symbols of mathematics for those who master the symbolic notation, and the handicap of any race that fails to develop the mathematics that it needs and to acquire an effective number system. All civilizations have developed mathematics of the same character and have discovered independently the same truths. The pupil should acquire an understanding of and interest in mathematics as one of the great achievements of man, the study of which if properly conducted develops power, mental alertness, initiative, and self-reliance. He should learn to recognize and enjoy the beauty of geometric forms in art, architecture, and nature.

As yet very little subject matter is presented in our educational literature for the purpose of helping the pupil definitely and directly to acquire the cultural values of mathematics. Some very interesting material is now being prepared under the auspices of the American Council of Education.[17] It furnishes informational readings to be used in connection with courses in elementary schools and high schools and aims to stimulate the pupil to further independent and extensive reading.

Measuring the development of functional thinking.—It is readily conceded that the ability to think functionally is a major objective

[17] Committee on Materials of Instruction, *The Story of Numbers; The Story of Weights and Measures* (5835 Kimbark Ave., Chicago).

of mathematics. So important is the function concept that mathematics has sometimes been defined as "the science of serial, spatial, quantitative, and magnitudinal relations." Although for thirty years leaders in mathematics have urged emphasis in teaching on functional thinking, it cannot yet be said that the idea of functional thinking has been made an organic feature of mathematical instruction. Only in some of the more recent textbooks is the topic beginning to receive a fair degree of attention. Recently Lennes examined eight elementary algebras that were published within the last five years, and came to the conclusion that "the ideas of function and of variation are introduced in a perfunctory way which cannot leave any lasting impression on the minds of the students. . . . . In no case can functionality or variation be said to be more than one of a large number of topics studied in algebra."[18] Functional thinking seems to be an objective which teachers and textbook-writers find difficult to attain.

The method of solving the problem should be similar to that suggested in the foregoing pages for the study of the cultural objective of mathematics. An examination of the definitions of functional thinking will disclose the characteristics of the concept. For the mathematics of the secondary school the function is commonly defined as *an expression containing a variable, as x, which assumes a definite value when a numerical value is assigned to x.* The characteristics of functional thinking implied in this definition are the understanding of the ideas of variation and variable, the recognition of dependence and other relationship which exists between variables, the correspondence of a value of one variable to a definite value of another, the ability to express relationships in mathematical symbols, and an understanding of the effect of changes in a variable upon related variables.

An examination of textbooks will disclose ways of representing variation, dependence, and correspondence, as in verbal statements, in tables, in graphs, and in formulas. It will suggest concrete materials that may be used for purposes of instruction and testing. It will indicate where the courses offer opportunity for teaching the concept.

[18] J. Lennes, "The Function Concept in Elementary Algebra," *Seventh Yearbook: National Council of Teachers of Mathematics* (Columbia University, 1932).

The published articles of writers and the textbooks on the teaching of mathematics contain additional materials showing the importance of the function concept not only in mathematics but also in the everyday life of children and adults. An analysis of the activities of children has shown that their experiences with functional thinking begin in the lowest grades of the elementary school. Unfortunately these experiences are seldom utilized by teachers to develop functional thinking. Thus, the pupil learns early that if he can buy 2 squares of candy for 5 cents he will be able to buy half a dozen for 15 cents. The number of squares he may buy depends on the number of cents in his purse. Precisely, the number of squares is twice the number of nickels, a fact which he later learns to express algebraically by the formula $m=2n$. In the upper grades of the elementary school many rules are taught which have functional aspects and may be used to train pupils in functional thinking.

In junior high school mathematics quantitative relationships are encountered almost constantly. The theorem about the sum of the angles of a triangle may be presented as a relation between the three angles, which enables the pupil to determine one of them as soon as the other two are known. The theorem of Pythagoras is an expression of the relation between the three sides of a right triangle. The perimeter of a triangle depends upon the lengths of the sides, and the circumference upon the radius or diameter of the circle. Areas and volumes depend upon known dimensions. An exceedingly large part of junior geometry deals with formulas and constructions in which dependence, correspondence, and relationships are prominent features. The course offers an abundance of opportunities for training in functional thinking.

In the senior high school the function should become the central theme of mathematics. Klein of Germany calls it the "soul" of mathematics. Tannery of France says that without the function concept the student has not the slightest idea of what mathematics really is. If he fails to understand it, he does not understand mathematics, no matter how thoroughly he is drilled and how proficient he may be in manipulating symbols. The National Committee of 1923 urged teachers to have the idea of function constantly in mind and to direct the pupil's advancement consciously along the lines which will present first one and then another of the ideas upon

which finally the formation of the general concept of functionality depends.

Tests should be developed to measure attainment of the various aspects of functional thinking. They will enable teachers to see these aspects clearly in their own minds, to study methods of teaching them, and to improve the methods until the desired outcomes are attained with the pupils. The problem of measuring the development of functional thinking involves the following steps:

1. To analyze the meaning of the concept to determine its characteristics.

2. To devise tentatively test items to show whether the various characteristics are being attained by the pupils.

3. To administer the tests to small groups of pupils of various school levels for the purpose of finding suggestions for improving the statements of the test items, of determining time limits, of eliminating items that are too difficult, and adding new items that should be included.

4. To revise the tests and to administer them again to larger groups. The results should be used to make further revisions.

5. To have the tests mimeographed in the final form in which they are to be administered to all pupils.

Typical illustrations of test items that may be used to measure various characteristics of functional thinking follow. A more detailed discussion will be found in the *Seventh Yearbook*.[19]

I. Recognizing dependence
  a) When stated in verbal form
    1. The amount of pay a man receives depends on the number of .... he works and his .... per day.
    2. The perimeter of an equilateral triangle depends on the length of the ....
    3. The distance a train travels at uniform rate depends on the ....
  b) When stated in well-known formulas
    1. $A = \frac{1}{2}bh$ shows that the area of a triangle depends on...
    2. $i = .04pt$ shows that the income received from an investment depends on..

[19] E. R. Breslich, "Measuring the Development of Functional Thinking in Algebra," *ibid.*, chap. v.

3. $S=(n-2)180$ shows that the sum of the angles of a polygon depends on..

....

II. Expressing relationships in algebraic symbols

a) When stated in verbal form

Translate into symbols

1. John is 5 years younger than Henry.
2. Paul's father earns 3 times as much money as he.
3. The speed of a train exceeds the speed of an automobile by 10 miles.

b) When the variables are named

Express the relation between

1. Cost of railroad ticket and distance of a trip at 3 cents a mile.
2. Centigrade and Fahrenheit.
3. Time and distance of falling object.

c) When stated in tables

Express as an equation the relationship in

1.

x	1	2	3	4	5	6
y	3	6	9	12	15	18

2.

x	1	2	3	4	5
y	12	6	4	3	2.4

3.

x	1	2	3	4	5
y	3	5	7	9	11

III. Understanding the change in a variable caused by a change in a related variable

a) When the relation is expressed by a table

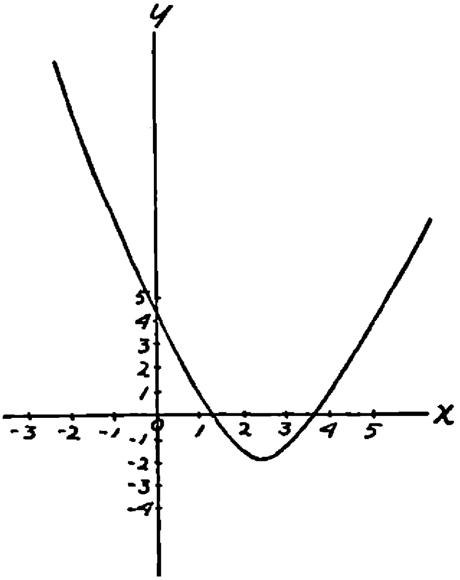
If $t=0$	3	5	6	9	10
then $d=0$	120	200	240	360	400

From the table it follows that

1. If t increases then d ....
2. If t decreases then d ....
3. If t is doubled then d ....
4. If t is divided by 2 then d ....

b) When the relation is stated by a formula
In the formula $c = 3.4d$
1. If d is doubled then c ....
2. If d is divided by 3 then c ....
3. If d is increased by 5 then c ....

c) When the relation is shown by a graph
1. If x increases from 0 to 2 then y ....
2. If x changes from 3 to 5 then y ....
3. If x decreases from 0 to -3 then y........
....



IV. Interpreting relationships
a) When stated by a formula
1. In $v = abc$ how does v change if a and b are constant and c is doubled?
2. In $v = abc$ how does c change if v and a are constant and b is doubled?
3. How does $\frac{k}{x}$ vary if k is constant and x increases?

b) When stated in graphs
1. From the statistical graph (e.g., temperature) tell when it is warmest; coldest; when the rise (or fall) is greatest; when the temperature is 92°.
2. From the graph of $y = 3x - 7$ tell the value of y when $x = 2$; the value of x when $y = 0$; the root of $3x - 7 = 0$.
3. From the graph of $y = x^2 - 5x + 6$ tell the value of $x^2 - 5x + 6$ when $x = 2$; the roots of the equation $x^2 - 5x + 6 = 0$; the axis of symmetry; the minimum value of $x^2 - 5x + 6$.

c) When expressed in equations
   1. If in the equation $12x+18=15x+6$ the left side is changed to $12x+12$ what is the right side?
   2. If the left side is changed to $4x+6$ what is the right side?

V. Recognition and understanding of functions
   a) When they occur as polynomials
      1. Evaluate $2x-7$ for $x=0 \ldots 10$.
      2. When is $2x-7$ equal to zero?
   b) When they are of the form $ax+b$
      1. Show that v_0+at is of the form $ax+b$ since $a=$............, $b=$..........
      2. Show that $s_0+\frac{1}{2}gt^2$ is of the form ax^2+bx+c since $a=$............, $b=$............, $c=$....................
   c) When they occur in equations
      1. Show that $2\pi r=84$ is of the form $ax=b$ since $a=$............, $b=$............
      2. Show that $3x^2-5x=1$ can be changed to the form $ax^2+bx+c=0$ where $a=$............, $b=$............, $c=$............

VI. Variation
   a) Express in symbols
      1. The price of sugar varies as the weight:....................................
      2. The area of a circle varies as the square of the radius:................ ....
      3. The time of traveling a given distance varies inversely as the rate.
   b) Express in the language of variation

1. $i=.05p$	3. $v=\frac{4}{3}\pi r^3$
2. $s=\frac{1}{2}gt^2$	4. $pv=K$

Although the foregoing test was only in tentative form when administered, it disclosed important results in regard to the development of functional thinking:

1. Understandings of nearly all of the characteristics of functional thinking are acquired early by certain pupils. With others they are developed slowly. Still others do not acquire them at all.

2. Entire groups attain satisfactory results with some of the characteristics, e.g., with interpreting statistical graphs, recognizing dependence in verbal statements of relationships, and recognizing changes in tables of numerical facts.

3. Most pupils when entering the secondary-school period possess considerable knowledge about some phases of the function

concept. They retain and extend this knowledge as they continue in school work.

4. It will be necessary to improve the methods of teaching the aspects of functional thinking in which the results were unsatisfactory but which are of importance to all pupils, e.g., recognition of changes in equations, understanding of relationships expressed in formulas and graphs, and knowledge of the meanings of linear and quadratic functions.

Summary.—It was said in the foregoing pages that a study of objectives should lead to improvement in teaching. Information and skills have been too often the major objectives of mathematics, and sometimes they have been the only objectives. Equally important, if not more important, is the acquisition of such objectives as power, appreciation, understanding, habits, and attitudes. The teacher should not expect that they will be attained accidentally as by-products. He should provide for them in his teaching as carefully and as definitely as for the acquisition of information and skills. Such a program calls for the development of tests which may be used to measure the extent to which the objectives are being attained with pupils and which lead to improved methods of teaching.

BIBLIOGRAPHY

Alice Irene, Sister. "Some Objectives To Be Realized in a Course in Plane Geometry," *Mathematics Teacher*, XXII (December, 1929), 435–47.

Allen, Gertrude. "Objectives in Teaching of Mathematics in Secondary Schools," *ibid.*, XVI (February, 1923), 65–78.

Bobbitt, Franklin. *Curriculum Making in Los Angeles*, "Supplementary Educational Monographs," No. 20, chap. vii. Department of Education, University of Chicago, 1922.

———. *How To Make a Curriculum.* Boston: Houghton Mifflin Co., 1924.

Boyce, George A. "Applying to Mathematics the Modern Ideas of Educational Science," *Nation's Schools*, IV (October, 1929), 67–71.

Breslich, E. R. "The Development of Functional Thinking," *Seventh Yearbook: National Council of Teachers of Mathematics*, chap. v. New York: Columbia University, 1932.

———. *The Technique of Teaching Secondary-School Mathematics*, chap. viii. Chicago: University of Chicago Press, 1930.

Cardinal Principles of Secondary Education. U.S. Bureau of Education, *Bull. 35* (1918).

Davis, Alfred. "Valid Aims and Purposes for the Study of Mathematics in Secondary Schools," *School Science and Mathematics,* XVIII (February, 1918), 112–23; *ibid.* (March, 1918), 208–20; *ibid.* (April, 1918), 313–24.

Edwards, William H. "The Need of Testing Practices in Curriculum Revision," *Mathematics Teacher,* XXII (October, 1929), 320–22.

Georges, J. S. "Mathematics in the Scheme of General Education," *School Science and Mathematics,* XXXI (December, 1931), 1056–67.

———. "A Study of Procedures Used in Determination of Objectives in the Teaching of Mathematics," *Mathematics Teacher* (March, 1929), 156–66.

Hassler, J. O., and Smith, R. R. *The Teaching of Secondary Mathematics,* pp. 105–30, 196–200, 224–38, 297–316. New York: Macmillan Co., 1930.

Jackson, Dunham, and Others. "Report of the Committee on Geometry," *Mathematics Teacher,* XXIV (May, 1931), 298-302.

Judd, Charles H. "Informational Mathematics versus Computational Mathematics," *ibid.,* XXII (April, 1929), 187–96.

Kempner, Aubrey J. "The Cultural Value of Mathematics," *ibid.,* pp. 127–45.

Kinney, J. M. "The Value of the Verbal Problem," *School Science and Mathematics,* XXV (March, 1925), 267–70.

Lennes, N. J. "The Function Concept in Elementary Algebra," *Seventh Yearbook: National Council of Teachers of Mathematics,* chap. iii. Columbia University, 1932.

McCain, A. B. "Aims, Purposes and Methods in High School Trigonometry," *School Science and Mathematics,* XXXI (May, 1931), 516–24.

Minnick, J. H. "The Aims of Mathematical Education," *Mathematics Teacher,* XIV (October, 1921), 297–304.

———. "Mathematical Tests: Their Relation to the Mathematics Teacher," *ibid.,* XI (June, 1919), 199–205.

Moore, R. N. "Objectives in Mathematics," *High School,* VI (April, 1929), 105–8.

National Council of Teachers of Mathematics, The. *Second Yearbook.* New York: Teachers College, Columbia University, 1927.

National Society for the Study of Education, The. *Twenty-sixth Yearbook.* Bloomington, Ill.: Public School Publishing Co., 1926.

North Central Association. "Report of the Subcommittee on Junior High School Mathematics," *North Central Association Quarterly,* II (March, 1928), 9–32.

Nyberg, Joseph A. "Objectives in Intermediate Algebra," *Mathematics Teacher*, XX (December, 1927), 451–59.

Perry, Winona. "What Are the Real Values of Geometry?" *ibid.*, XXI (January, 1928), 51–54.

Rankin, W. W. "The Cultural Value of Mathematics," *ibid.*, XXII (April, 1929), 215–23.

Reeve, W. D. "Objectives in Teaching Demonstrative Geometry," *ibid.*, XX (December, 1927), 435–50.

———. "Objectives in Teaching Intermediate Algebra," *ibid.*, March, 1927, pp. 150–60.

———. "Objectives in the Teaching of Mathematics," *ibid.*, XVIII (November, 1925), 385–405.

Schorling, Raleigh. *A Tentative List of Objectives in the Teaching of Junior High School Mathematics with Investigations for the Determining of Their Validity.* Ann Arbor, Mich.: George Wahr, Inc., 1925.

Shanholt, Henry H. "The Objectives of Algebra," *High Points*, XIV (February, 1932), 60–63.

Walker, Helen M. "What the Tests Do Not Test," *Mathematics Teacher*, XVIII (January, 1925), 46–53.

Williams, L. A. "Analysis Techniques in Curriculum Making," *School Review*, XLI (June, 1933), 437–42.

Wood, Walter H. "Some Reflections on the Current Discussion of Secondary Mathematics," *School Science and Mathematics*, XVII (December, 1917), 815–18.

CHAPTER VI

METHODS OF SELECTING MATERIALS FOR TEACHING PURPOSES

Criticisms of mathematics as traditionally taught.—The mathematical subjects were among the first to be admitted to the high-school curriculum. For a long period their position was considered secure and well established. They were able to retain this position because of their supposedly great mental-training value and because of the usefulness of mathematics. More recently, when the newer subjects—the sciences, arts, commercial studies, and others—began to compete with the older subjects for places in the curriculum, the security of position of the older and well-established departments was endangered. When schools began to make scientific studies of the results of teaching, and when it was found that an abnormally large number of pupils failed to profit sufficiently from the study of mathematics to make passing grades, dissatisfaction with the courses and with the methods of teaching them continued to grow. It is not uncommon to find percentages of failures in algebra and geometry varying from 15 to 35 per cent, and sometimes they are higher than 35 per cent. Thorndike concluded from the results of his study that the student with an intelligence quotient below 105 would be unable to complete successfully traditional Freshmen algebra. He estimated that this applies to about 56 per cent of the high-school Freshmen group. Orleans[1] reports that the percentage of failure in geometry in the New York City schools ranges from 21 to 36 per cent, and that more than three fifths of those who pass receive a barely passing mark.

The results of examinations such as those given by the College Entrance Examination Board are equally discouraging. Usually they show a mortality exceeding 25 per cent, although most pupils make special preparation before they attempt these examinations. It is not surprising, therefore, that parents have severely criticized

[1] Joseph B. Orleans, "The Fusion of Plane and Solid Geometry," *Mathematics Teacher*, XXIV (March, 1931), 155.

the conventional courses in algebra and geometry and have questioned their educational values.

Further criticisms have come from instructors in subjects other than mathematics in which the pupils are required to make use of their mathematical knowledge. They claim that weakness in mathematics contributes heavily to the mortality in their subjects.

In this contention they are supported by instructors in college who teach mathematics to pupils who made passing grades and received credit for courses in secondary-school mathematics. Their complaint is that the pupils have not retained enough of the subject to use it successfully in later courses in mathematics.

To the foregoing criticisms may be added those of the business men who claim that high-school graduates are not sufficiently grounded in the fundamentals of arithmetic used in ordinary business transactions, and of the men in the industries who find them lacking in the mathematics necessary for success in industry.

Teachers who are satisfied with the prevailing type of mathematics are inclined to wave aside most of the criticisms as not being based on facts or as highly exaggerated. However, in the examinations of the College Entrance Examination Board and of the other examining bodies whose records are available for inspection the evidence of failures of pupils is not denied. If the examinations were administered to pupils who have made no special preparation for the examination the mortality would be far greater. Nevertheless, teachers may refuse to be disturbed by the results by claiming that the validity of the examinations is questionable. However, the results obtained with simpler examinations than those of the College Entrance Board are anything but reassuring. Thorndike[2] gave a very simple test in algebra in a number of schools ranking above the average. The results were so poor that he came to the conclusion that the pupils after studying algebra for one year had attained mastery of nothing whatsoever. Other investigations have verified his findings.

Needs for careful selection of mathematical materials.—It is futile to deny that there is need for readjustments in the content and methods of secondary-school mathematics. One of the most drastic recommendations has been the reduction of the mathematical re-

[2] E. L. Thorndike, *The Psychology of Algebra* (Macmillan Co., 1923), p. 320.

quirements for graduation from the high school. Originally schools required two and one-half years of mathematics. During the present century, however, there has been a decline in mathematics as a prerequisite for college entrance and high-school graduation. The requirement has been cut down to the point where many high schools require but one year, and where some have entirely eliminated the requirement. In some states the departments of education have ruled that not even a unit of mathematics should be required of approved high schools.

The same view is reflected in the arguments of those who contend that the average citizen does not need to study secondary-school mathematics as a separate subject and that his needs should be provided for incidentally in connection with other school subjects. The following bulletin sent out by the Los Angeles city schools reflects the attitude of those responsible for the mathematical curriculum in that system:

> It is probable that no department of the high school has a more difficult or baffling problem of reorganization of its activities than the mathematics department. Let us illustrate.
>
> The city has decided that commercial students going into business do not need algebra, geometry, or trigonometry for general, cultural, or disciplinary training.
>
> Since this is a large and representative group of students, it appears to follow, if this decision is correct, that students in general do not need algebra, geometry, or trigonometry for general, cultural, or disciplinary training.
>
> The city has decided that commercial students need full and intensive training in the mathematics of their vocation.
>
> This probably typifies the need of every vocational group. It needs full and intensive training in the mathematics of its vocation. But the mathematics will differ greatly from vocation to vocation and must be administered, therefore, according to the special needs and as a part of the vocational training.
>
> The vocational mathematics for commercial students is administered in this city as a vocational course in the commercial department.
>
> This appears to represent the proper placement of all vocational mathematics courses, not in the general department of mathematics, but in the appropriate vocational department.[3]

[3] Editorial, *School Review*, XXX (June, 1922), 407–8.

On the foregoing bulletin the editor makes the following comments:

It is the belief of the writer that these decisions of the city as regards the mathematics of the commercial students are educationally correct and that the deductions that appear naturally to follow are educationally correct.

If this is true, then algebra, geometry, and trigonometry have justifiable places in the curriculum only when they are necessary portions of vocational courses; and in such cases the specific content is differently dictated by different vocations.

These statements are equally applicable to those who finish their schooling with the high school and to those who take additional years of work in college. Neither the length nor the place of one's training dictates one's needs.

In the foregoing statements there is one possibility of error. It may be that commercial students *do* need the disciplinary values of algebra and geometry, but that they must forego them because of the exigencies of the time schedule; that those who take the longer training of both high school and college need not forego them and therefore may secure the disciplinary values. There is, however, no proof of the disciplinary values. The general intelligence quotient does not seem to be raised by a study of algebra and geometry; and one, therefore, is not given greater power to think in general outside of the mathematical fields.

The college-entrance demand is largely dictated by the disciplinary hypothesis—without proofs. In far larger measure, however, it is held to *for selective purposes*. It is not that the students need algebra and geometry, but that the colleges need a selected body of students, and algebra and geometry have been, aside from classic languages, until recently, the best selective devices. So long as the colleges demand them for this purpose, the high schools must administer them for this purpose. They should know, however, that they are doing it for the good of the colleges and not because they are demonstrably serving their students.

The usefulness of much of the content of the mathematical courses in future occupations, in professions, and in other school subjects has been questioned by leaders in mathematics. Young[4] believes that the average citizen has little need of mathematical facts and that the opportunity to use them beyond the merest elements of arithmetic is small.

[4] J. W. A. Young, *The Teaching of Mathematics*. New York: Longmans, Green & Co., 1924. Pp. 13.

Geometry is not studied, and never has been studied, because of its positive utility in commercial life or even in the workshop. In America we commonly allow at least a year to plane geometry and a half year to solid geometry; but all of the facts that a skilled mechanic or an engineer would ever need could be taught in a few lessons. All the rest is either obvious or is commercially and technically useless.[5]

Not one tenth of the graduates of our high schools ever enter professions in which their algebra and geometry are ever applied to concrete realities; not one day in three hundred and sixty-five is a high-school graduate called upon to 'apply,' as it is called, an algebraic or a geometrical proposition. . . . . Why, then, do we teach these subjects, if this alone is the sense of the word "practical"! . . . . To me the solution of this paradox consists in boldly confronting the dilemma, and in saying that our conception of the practical utility of those studies must be readjusted, and that we have frankly to face the truth that the "practical" ends we seek are in a sense *ideal* practical ends, yet such as have, after all, an eminently utilitarian value in the intellectual sphere.[6]

Claims that the content of the mathematical courses has been poorly selected have been verified by investigators who have prepared statistics which make it appear that the system is lacking in purpose, is unreal to the pupils, and remote from the world they live in. Thorndike and Woodyard made an analysis of over forty textbooks from which they reached the conclusions:

(1) In no courses except advanced mathematics is there need for skill in the manipulation of polynomials; (2) in present textbooks there is no use made of the mathematical concept of function; (3) the study of equations has no place of use except in chemistry, physics, and agriculture; (4) the making of formulas is hardly ever required of high school pupils; (5) the evaluation of formulas is required only in chemistry and physics, and in the chemical and physical parts of general science; (6) the mathematical graph practically does not occur in high school work; (7) the statistical graph is used more or less in all the subjects investigated.[7]

[5] D. E. Smith, *The Teaching of Geometry.* Boston: Ginn & Co., 1911. Pp. 7.

[6] T. J. McCormack, *Why Do We Study Mathematics?* Cedar Rapids, Iowa, 1910. Pp. 9.

[7] E. L. Thorndike and Ella Woodyard, "Uses of Algebra in Study and Reading," *School Science and Mathematics*, XXII (May and June, 1922), 405–12, 512–22.

The foregoing discussion may be summarized briefly as follows: The mathematics of the secondary school owe the position which they occupy in the curriculum to the historical development of education. With the changes in the new education much dissatisfaction developed with the content and methods of teaching mathematics. Criticisms have come from parents, investigators, business men, and from leaders in education and in the subject of mathematics. Recommendations have been made to reduce the mathematical requirements for graduation from the high school and to replace obsolete and useless content by new material related to the needs of the pupils and the community.

How algebra and geometry found places on the secondary-school curriculum.—Before attempting to reorganize courses as well established as traditional algebra and geometry, one may ask whether originally the content of these courses was the outcome of a plan designed to meet the needs and abilities of the pupils in the schools in which these subjects are now required for graduation. The answer to the question may be seen from information that may be gathered from catalogues and school reports of the early American schools.

It appears from Table XV that the location of algebra and geometry in the secondary-school curriculum is quite recent. The development of the present secondary mathematical curriculum in America began during the first part of the eighteenth century. The earliest American colleges were modeled after the English universities, and the courses of study were at first taken from English institutions. Furthermore, instruction was given by men who either came from England and Scotland or had gone to England to study. Accordingly, the American colleges took over courses in algebra and geometry which were written for students in English colleges and offered them to their own students for study as logical sciences. It appears that the development and organization of algebra and geometry were based upon the disciplinary point of view. They were planned as college subjects.

Table XV shows that in the year of 1700 algebra was not yet offered in American colleges. It was studied very little before the middle of the eighteenth century, but was gradually gaining a foothold. The place of algebra and geometry in the courses offered by

the first colleges, as disclosed by an examination of college catalogues and as shown in Table XV, was generally in the Senior year, but both courses were gradually moved downward. Usually algebra was studied prior to geometry. Ultimately it was given a place in the Freshman courses of the colleges, geometry being offered in the Sophomore course. However, in the downward movement no reorganization seems to have taken place.

TABLE XV

THE PLACE OF ALGEBRA AND GEOMETRY IN THE CURRICULUM OF THE EARLY COLLEGES

Year	Subject	College	Class
1719	Geometry	Yale	Senior
1722	Algebra	William and Mary	....
1726	Algebra	Harvard	....
1742	Geometry	Yale	Sophomore
1743	Algebra	Yale	Freshman
1750	Algebra	Pennsylvania	....
1755	Algebra	Columbia	Sophomore
1755	Geometry	Columbia	Junior
1756	Algebra, Geometry	Pennsylvania	Freshman, Sophomore
1770	Algebra, Geometry	New Jersey	....
1778	Algebra	Harvard	Sophomore
1785	Algebra	Columbia	Freshman
1787	Geometry	Harvard	Sophomore
1802	Algebra, Geometry	Princeton	Junior
1811	Algebra	Dartmouth	Sophomore
1818	Geometry	Harvard	Freshman
1834	Algebra	Dartmouth	Freshman
1835	Algebra	Harvard	Freshman
1840	Algebra	Princeton	Freshman
1840	Geometry	Princeton	Sophomore

The growing high school gave a new impetus to the study of mathematics and gradually placed algebra and geometry on the curriculum. In 1820 algebra was made an entrance requirement for Harvard, geometry becoming a required study in 1844. Thus the preparatory schools took over the courses in algebra and geometry, but presented them almost as they had been taught in the colleges. The old sequence was also continued—algebra before geometry. At first they were taught in the upper classes, but gradually they were moved down to the first and second years of high school. Table XVI shows that after the first half of the nineteenth century

algebra was generally placed in the first year of the high school in the eastern schools and geometry in the second. Since then they have remained in the same position.

Thus, the traditional order—arithmetic, algebra, geometry, and trigonometry—is inherited by the high school from the college. The disciplinary objective of the college was accepted as dominant by

TABLE XVI

THE PLACE OF ALGEBRA AND GEOMETRY IN THE CURRICULUM OF THE EARLY SECONDARY SCHOOLS

Year	Subject	School	Class
1799......	Geometry	Phillips Exeter Academy	....
1814......	Algebra, geometry, trigonometry	Boston Public Latin	....
1818......	Algebra, geometry	Phillips Exeter Academy	Third, fourth
1821......	Algebra, geometry, trigonometry	English Class. (Boston)	Second, third
1824......	Geometry	Worcester Female High	....
1824......	Algebra, geometry	New Haven Seminary	....
1826......	Algebra, geometry	Boston Girls' High	Second, third
1828......	Algebra, geometry	Worcester Female High	....
1828......	Algebra, geometry	Academies of New York	....
1845......	Algebra, geometry	Worcester Class. High	Second, fourth
1850......	Algebra, geometry	Middletown High	Third, fourth
1852......	Algebra, geometry	Philadelphia High	First, second
1852......	Algebra, geometry	Toledo, Ohio	First, second
1854......	Algebra, geometry	Bowditch (Salem, Mass.)	Second, third
1862......	Algebra, geometry, trigonometry	Worcester, Mass.	First, second, third
1865......	Algebra, geometry	Hartford, Conn.	First, second
1865......	Algebra, geometry, trigonometry	Concord, N.H.	First, second, third
1867......	Algebra, geometry, trigonometry	Portland, Me.	First, second, fourth

the secondary schools. The sequence is in no sense the result of an attempt to create a suitable curriculum adapted to the needs and mental development of the secondary-school pupils of today. A different order is therefore worthy of serious consideration.

There has been considerable activity among the teachers of mathematics to improve the courses taught in the secondary schools. Committees have been at work formulating statements of the aims and values of mathematics. Attempts have been made to vitalize the subject by emphasizing practicality and social uses.

Reports, articles, and books have appeared on reorganized and reconstructed mathematics. Reforms have been made, and they have come largely from the teachers themselves. Evidence as to improvement is easily obtained by a comparison of the textbooks of today with those of thirty years ago. However, it is in keeping with the facts to say that the changes should be more far reaching to make algebra and geometry successful school subjects. What is needed is a thoroughgoing, radical reconstruction.

Important movements aiming to improve the content of mathematics.—It should not be inferred that there has been a lack of leaders with vision. As early as thirty years ago men like Perry and Nunn in England; Tannery, Borél, and Laisant in France; Klein in Germany; and E. H. Moore in America have pointed the way toward improvement in unmistakable terms. Their views have been readily accepted by the teachers of mathematics. The suggestions made by them have been discussed widely in teachers' conferences and mathematical journals; but textbook-makers and teachers have been slow to put the most important of the recommended changes into actual practice.

Perry's recommendations.—During the years 1900–1902 John Perry[8] delivered a number of lectures in England on the teaching of mathematics. He advocated: (1) that steady emphasis be placed on the practical uses of mathematics as found in mechanical drawing, problems of physics, chemistry and enginneerig, and on graphical methods, so as to make an appeal to the pupil through the usefulness and practicability of the mathematical subjects; (2) that the study of useful mathematical principles should not be deferred to the time when logical proof could be established, and that the truths of many propositions in geometry be assumed without proofs to "let the young boy get quickly to the useful part of mathematics"; (3) that understanding of many of the general principles of mathematics be developed by experimentation. "The pupil should be taught through his own experiment, through concrete examples worked out by him," since often this may be preferable to proof by logic. "Men who teach demonstrative geometry and

[8] "The Teaching of Mathematics," *Educational Review*, XXIII (1902), 158–81.

orthodox mathematics generally . . . . are producing a dislike and hatred . . . . and are doing incalculable harm."

Perry's views were readily indorsed by the teachers in America, who felt that much was to be gained by connecting mathematics with other subjects of interest to the boys and girls. A number of papers have been published which show how the work in mathematics may be closely correlated with such school subjects as physics and the other sciences. An extreme attitude was reflected in the arguments of those who advocated that the formal study of mathematics be abandoned and that the subject be developed "incidentally" in other subjects through applications. One direct outcome of the Perry movement was the appearance of numerous applications in textbooks. Later experience has shown that many of the so-called real or practical applications were not as real to the pupils, especially to the girls, as some of the traditional problems which they replaced. Moreover, the applications were often more difficult than the mathematical laws that were to be illustrated. However, the influence of the Perry movement which stressed usefulness in teaching mathematics left its mark on the content of secondary-school mathematics not only in England and America, but also in France and Germany. Undoubtedly, a large amount of the traditional subject matter has little practical value, and the teaching of applications along with the pure mathematics does much to make the study interesting and worth while to the student. There is need of constant searching among the applications of mathematics for the purpose of selecting those that are real to the pupil and that make possible sufficient experience with the processes. If the material is brought in from adult life, it should be something that has real meaning to the pupil. Preferably mathematical material should be taken from the domain of other school studies such as general science, geography, physics, and chemistry.

Emphasizing concrete subject matter.—Algebra and logical geometry are full of abstractions that are exceedingly difficult for the learner. Hence, mathematicians in France, Germany, England, and America have urged strongly the replacing of much of the abstract by the concrete. They have advocated that the abstract relations of algebra and geometry be derived from concrete experi-

ences in order that the material be thoroughly understood and mastered by the pupil and therefore become available as a basis for future work. The practice of beginning courses in algebra and geometry with pages of abstract definitions, axioms, rules, and processes has been questioned as of doubtful value. Historically, mathematics has not been developed that way. Concrete mathematics begins with observation and experimentation. Instead of having mathematics presented to him in its abstract form, the pupil is led to make his own abstractions. The intuitive method is accepted as a mode of proof. It makes use of models, ruler, protractor, compasses, balances, and squared paper.

The effect of the movement for concreteness was felt most in geometry where it has led to the laboratory method and to the admission of a larger number of fundamental assumptions than are commonly found in texts using Euclid as a basis. In algebra the movement has brought about the introduction of much geometric material with which to make the abstract laws and processes concrete. There is a tendency to select problems that are related to the pupil's experiences, activities, and interests.

Bringing the fundamental notions of advanced courses into the lower courses.—Much of the material contained in the traditional courses which formerly occupied a prominent place in teaching is of no immediate value to the pupil, and much of it he will probably never use. On the other hand, many fundamental and simple but exceedingly valuable notions that are developed in the upper courses are kept from him unless he continues the study of mathematics until he has fulfilled the prerequisites for these courses. Hence, it has been recommended by leaders in mathematics that, by postponing to a later stage a considerable body of unimportant subject matter of elementary algebra and demonstrative geometry and by eliminating all materials that are obsolete or useless, room may be made for some of the simple but valuable work usually offered only in the advanced courses of the high school and junior college.

To make possible the early introduction of the elements of the upper courses, the French mathematician, Laisant,[9] suggested that arithmetic, algebra, and geometry be "freed from the multitude of parasitic propositions and reduced to the exposition of directive

[9] C. A. Laisant, *La mathématique.* Paris: Carré & Naud, 1898. Pp. 270.

ideas and essential methods. . . . . Not only will valuable time have been gained but also greater clearness of ideas imparted. This will permit the introduction of the elements of analytic geometry and calculus."

Tannery and Borél, of France, advised that some of the objectionable proofs of solid geometry be replaced by common-sense discussions based on models, and that the time thus saved be used for teaching the elements of analytic geometry and the elementary notions of differential and integral calculus. Tannery recommended the teaching of some calculus, preceding the study of surfaces and solids. After eight or ten lessons of analytic geometry and integral calculus, "a quarter of an hour will be sufficient for the development of all the formulas for the volumes of elementary geometry."[10] He advised that the reasoning which is customarily used to establish the equivalence of volumes of oblique and right prisms "be kept in an historical museum as evidence of how intelligent our ancestors were."

The same idea was expressed by Perry: "If we begin by assuming more complex things to be true we shall get the same intellectual training with more knowledge."[11] He asserted that the elements of calculus can be comprehended by high-school students as easily as some of the things they are now required to learn. "Why not put aside ever so much of the work now taught, so as to let a young boy get quickly to the solution of partial differential equations and other useful parts of mathematics that only a few men now ever reach?"

Nunn has said that "notions which form part of the doctrines of calculus may be introduced at an early stage and developed side by side with other algebraic ideas."[12]

Klein, of Germany, believed that differential and integral calculus in its beginning is no more difficult than the difficult geometric proofs, constructions, and trigonometric relations.

Apparently the foregoing recommendations agree as to the desir-

[10] J. Tannery, "L'enseignement géométrique," *Review Pédagogique,* XVIII (1903), 1–27.

[11] *Op. cit.*, p. 170.

[12] T. P. Nunn, *The Teaching of Algebra.* London: Longmans, Green & Co., 1919. Pp. 21.

ability of bringing into the lower courses simple materials that traditionally have been taught in trigonometry, analytic geometry, and calculus. However, those who undertake to introduce this reform should keep in mind that materials must not be moved downward bodily without changes. Thus, attempts to teach traditional algebra and geometry in the seventh and eighth grades without modification will fail because these subjects have been found too difficult for older pupils. The treatment, terminology, and character of trigonometry and analytic geometry when taught in the secondary school cannot be the same as when these subjects are taught to college students.

Elimination of instructional materials.—Acceptance of the views expressed in the foregoing pages was voiced by the National Committee on Mathematical Requirements. It indorsed a practice which had prevailed among the progressive teachers of mathematics for some time by recommending eleven topics either for omission from the lower courses or for postponement to later courses:[13]

1. Highest common factor and lowest common multiple, except the simplest cases involved in addition of simple fractions.
2. Theorems on proportion relating to alternation, inversion, composition, and division.
3. Literal equations, except such as appear in common formulas, including the derivation of formulas and of geometric relations, or which show how needless computation may be avoided.
4. Radicals, except the following types: $\sqrt{a^2b}=a\sqrt{b}$, $\sqrt{\frac{a}{b}}=\frac{1}{b}\sqrt{ab}$, $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$.
5. Square root of polynomials.
6. Cube root.
7. Theory of exponents.
8. Simultaneous equations in more than two unknowns.
9. The binomial theorem.
10. Imaginary and complex numbers.
11. Radical equations, except such as arise in dealing with elementary formulas.

[13] *Reorganization of Mathematics in Secondary Education* (a report by the National Committee on Mathematical Requirements, 1923), p. 27.

For geometry the Committee recommended a general reduction of theorems and constructions.

The following additions are recommended for the course in secondary mathematics:

1. Numerical trigonometry for Grades VII–IX, including
   a) Definitions of sine, cosine, and tangent
   b) Their elementary properties as functions
   c) Their use in solving problems in right triangles
   d) The use of tables of these functions (to three or four places)
2. Elementary statistics for Grades X–XII
3. Elementary calculus for the twelfth grade including
   a) The general notion of a derivative as a limit
   b) Applications of derivatives to easy problems in rates and in maxima and minima
   c) Simple cases of inverse problems, e.g., finding distance from velocity, etc.
   d) Approximate methods of summation leading up to integration as a powerful method of summation
   e) Applications to simple cases of motion, area, volume, and pressure.

It is evident that the suggestions for additions can be carried out only if waste is eliminated wherever possible, if obsolete material is omitted, and if all of the instructional materials are organized in the most effective way. The organization as well as the method of presentation should be changed to adapt it to the ability of the pupil who is to study it. To illustrate, if numerical trigonometry is to be taught in a ninth-grade class the discussions must not be too technical and too difficult. It will be necessary to develop slowly and clearly the ideas of ratio of two numbers and ratio of two line segments before attempting to define the trigonometric ratios.

Methods used in selecting subject matter.—The foregoing pages have presented views which question the validity of much of the subject matter now taught in secondary-school mathematics. Three recommendations have been made for improving the content of the traditional courses: the elimination of topics no longer considered worth while; the removal to a higher level of topics for which there is no immediate need but which have deferred values in higher courses, in subjects other than mathematics, and in adult life; and the bringing-down to the lower courses of topics which traditionally have been taught at a high level but serve the indi-

vidual and social needs better than the materials they are to replace. The department of mathematics faces the problem of finding new materials which are to be introduced to improve the work. Obviously this means that the teachers cannot depend entirely on the textbook for the subject matter to be taught and that they must undertake research work to determine a body of new material or at least familiarize themselves with the findings of those who have contributed to the solution of the problem.

Various methods have been employed by investigators, but a study of the results makes it clear that no single method determines all of the subject matter to be used. However, each method supplements the findings of the others. To base recommendations on the results of one is not a safe procedure in the selection of instructional materials.

Analysis of textbooks.—Analyses of textbooks in mathematics may be made with the expectation of improving the content of courses by way of determining common practices, trends, difficulties, and relative emphasis on various topics. They show comparisons of what is being taught with what was taught formerly. They enable the teachers to know which recommendations of specialists and leading committees are being accepted by authors of textbooks. Teachers will thereby be aided in selecting instructional materials and in choosing the books they should use in their classes.

Objection to this method of selecting subject matter is made by those who condemn the textbooks as "notoriously unfit." It should be understood that the materials in textbooks are to be examined critically. For textbooks often contain only what has been taught and not always what "ought to be taught." On the other hand, the fact cannot be overlooked that the modern writer as a rule has made an intensive study of his field and that he aims to put into his books the best he knows. However, conservatism of his publishers may prevent him from including all of the materials that should be taught.

Several examples of textbook analysis were reported in detail in chapter iv and in another volume.[14] Dhus[15] made an analysis of

[14] E. R. Breslich, *Problems in Teaching Secondary-School Mathematics*, chap. ii.

[15] Mabelle D. Dhus, *A Determination of the Tendency of Junior High School Mathematics* (Master's thesis, University of Chicago, 1927).

junior high school textbooks and determined recent tendencies. Hunt[16] analyzed six third-grade arithmetics and obtained valuable information about the vocabulary needed in the subject. Humphrey[17] attacked the solution of the same problem for junior high school mathematics. Thorndike[18] studied the problem material of textbooks and determined the percentage of genuine problems in textbooks. The results of studies of the foregoing types are enlightening and give the teacher a great deal of information as to the kind of instructional materials he should choose for his classes.

Determination of mathematical materials needed in the study of subjects other than mathematics.—At all levels the courses in mathematics should include such mathematical information as is of greatest immediate value to the pupils. This means that the needs of subsequent courses in mathematics and of other subjects and the future social and vocational needs are not to receive too much emphasis. Moreover, they are not to be left entirely out of consideration. Much of the material which is valuable to the pupil of any particular grade will also be useful to him in other subjects, future courses, and the later activities of adult life.

The required information may be obtained by analyses of the textbooks used in the other courses and by inspection of the notebooks written by the pupils. The materials may be classified according to the mathematical subjects, as arithmetic in general science, or algebra in physics, or as mathematics in science and mathematics in physics. Several studies relating to the selection of materials will be reported in the following pages. Any teacher or department may easily verify the findings and may secure additional information by making similar analyses. Since arithmetic should be regarded an important phase of secondary-school mathematics, the findings as to the arithmetic used in various school subjects will be the first to be presented.

Arithmetic used in a seventh-grade course in sewing.—The course consisted of nine articles to be made by the pupils:

[16] Ava F. Hunt, *A Comparison of Vocabularies of Third-Grade Textbooks in Arithmetic and in Reading* (Master's thesis, University of Chicago, 1926).

[17] Cecil F. Humphrey, *The Vocabulary of Mathematical Textbooks in the Junior High School* (Master's thesis, University of Chicago, 1926).

[18] *Op. cit.*, p. 138.

1. Work apron
2. Tea apron
3. Night gown
4. Gown for pupil
5. Undergarment
6. Luncheon set
7. Curtain for assembly stage
8. Vanity set (circular)
9. Bungalow apron

The planning and designing of the articles required considerable use of arithmetic in connection with the following activities:

1. Estimating the cost of different materials required
2. Computing the cost of the article to be made
3. Determining the amount of time required to make the article
4. Tabulating the amounts of materials needed
5. Computing the circumference of a circle
6. Keeping an account of all expenses
7. Balancing accounts

The analysis of the notebook disclosed the following types of arithmetic required to enable the pupil to attain the solutions of the problems:

1. *Units of measure.*—The inch and yard were employed as units of length. Other units were dollars and cents.
2. *Roman numerals.*—Roman numerals were used in numbering drawings and problems.
3. *Size of numbers.*—The largest numbers used were four-figure numbers.
4. *Fractions.*—Decimal fractions were involved in computing costs. Common fractions and mixed numbers were used freely. The following were found: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{3}{16}$, $1\frac{1}{2}$, $1\frac{3}{8}$, $3\frac{1}{2}$, $12\frac{1}{2}$, and $37\frac{1}{2}$.
5. *Operations.*—The arithmetical operations were very simple. They involved adding lengths; multiplying lengths by whole numbers, as 2 or 3; simple subtractions; and finding areas of surfaces, as $27'' \times 30''$ or $\frac{3}{4} \times 20$.

Arithmetic used in a sheet-metal course for seventh-grade boys.—The facts were derived from analysis of pupils' notebooks which contained all the written work done in the course. As in the course in sewing, the amount of arithmetic used was simple. However, many errors were made by pupils which gave evidence that they were inaccurate in computing and that the processes were frequently not understood. The following was the arithmetic actually used:

1. *Units of measure.*—The inch was the only unit used in the course.
2. *Fractions.*—The fractions and mixed numbers found were: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{5}{8}$, $\frac{3}{16}$, $4\frac{1}{2}$, $10\frac{1}{2}$, and $8\frac{1}{4}$.

3. *Processes.*—The operations were mostly multiplications and divisions of small numbers. A few simple additions and subtractions occurred.

In making tin cups and funnels the pupils had to compute circumferences from given diameters which involved multiplication and division of integral numbers, and of common and decimal fractions. No other uses of arithmetic were found.

Arithmetic used in a manual-training course for seventh-grade boys.—In this course the pupils made a floor lamp, shoe box, stationary case, sewing cabinet, book rack, telephone table, and fernery. A considerable amount of arithmetical computation was required, but it was all of a very simple type.

The numbers were small, containing three figures or less. A few Roman numerals were found. The fractions actually used were: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$, $\frac{1}{16}$, $\frac{3}{8}$, $\frac{7}{8}$, and $\frac{3}{16}$; and the mixed numbers were: $1\frac{1}{2}$, $2\frac{1}{2}$, $5\frac{1}{2}$, $7\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{3}{4}$, $8\frac{3}{4}$, $11\frac{3}{4}$, and $20\frac{1}{4}$.

The operations were simple additions, subtractions, multiplications, and a few divisions. Whole numbers, fractions, and decimal fractions were involved in the operations.

The units of measure were the inch, foot, yard, square inch, square foot, degree, dollar, cent, and ounce.

Arithmetic used in seventh-grade general science.—The notebooks containing a complete year's work presented all the required written work. The amount of arithmetic actually used was small. Many large numbers were found in stating the distances of the planets from the sun and the sizes of their diameters. Hence, an understanding of large numbers seemed necessary. The numbers varied from one to eight places.

The operations of multiplication and division involved large numbers, sometimes containing as many as eight figures.

The units of measure were inch, mile; hour, minute, second; and degree.

Percentages, such as 20, 78, and 94 per cent, were used repeatedly.

Arithmetic used in general shop, a course for high-school boys.—The course comprised the making of the following articles: tin cup, funnel, wrought nail, center punch, staple, nail set, rivet set, clevis, chain links, gatehook, cold chisel, screw-driver handle, mallet, tool

tray, acid swab, soldering copper, scribber, ferrule, life-screw, screw-jack base, tap stock, and line cleat.

The whole numbers which occurred in the notebooks contained only one or two figures.

The fractions were $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, $\frac{3}{16}$, $\frac{5}{16}$, and $\frac{48}{64}$.

The mixed numbers were $1\frac{1}{2}$, $1\frac{1}{8}$, $1\frac{3}{4}$, $1\frac{3}{8}$, $2\frac{1}{4}$, $2\frac{3}{4}$, $2\frac{3}{8}$, $4\frac{3}{4}$, and $4\frac{3}{8}$.

The arithmetical operations were simple multiplications and divisions of whole numbers and fractions, such as are involved in finding the fourth term of a proportion when the other three are given.

Arithmetic used in a school print shop.—Numerous arithmetical problems arise in this work. They involve computations of the amount of type needed for a job, the number of pages required for a manuscript, the charges for printing, the number of words per square inch, the amount of type per page, the number of square inches in a page, the weight of type in a box, the value of type per pound, and discounts for cash.

The units used were the inch, pound, and square inch.

The fractions and mixed numbers were simple, such as $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{72}$.

The operations were mostly simple multiplications and divisions, such as 18×12, $42\frac{1}{2}\times42$, or $216\div8$, but some were more complicated, such as $(2\frac{2}{3}\times5\frac{1}{2})\div4$. Several of the problems required the extraction of square root.

Arithmetic used in a geography course.—Numerous important facts taught in geography need to be analyzed from the quantitative point of view. For example, in studying irrigation the pupil works with measures of distances, slopes, excavations, flow of water, and acreage. The following problems involving arithmetical work were typical:

Making comparisons by means of ratios of lengths, areas, populations, products, exports, and imports.

Finding relations between latitude, longitude, and time.

Computing cost, weight, and price.

Finding averages.

The course contained many large numbers used in connection with commercial facts, populations, exports, and imports.

The operations were simple additions, subtractions, multiplications, and divisions with whole numbers and fractions. Percentages were computed frequently.

Fractions usually had small denominators, as in $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{10}$, and $\frac{9}{10}$.

Decimal fractions were numerous.

The units of measure were many and varied, as pounds, miles, square miles, millions, and bushels. Standard units were used constantly.

Arithmetic used in a course in electricity offered in the ninth grade.—A great deal of arithmetical computation was found. There were numerous additions and subtractions of whole numbers and of such common and decimal fractions as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, 6.16, and 1.32.

Multiplications and divisions were frequent. The following are typical:

$$(1.32)\ (84),\ \frac{110.88}{1.52},\ \text{and}\ \frac{4{,}500–500}{4{,}500}.$$

Much arithmetic entered in problems of evaluating such formulas as

$$\frac{1}{R}E,\ \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3},\ \text{and}\ \frac{kl}{d^2}.$$

Decimal fractions were very important since the results were usually stated as decimal fractions.

Common fractions were changed to equivalent common fractions and to decimal fractions.

Decimal fractions were changed to common fractions.

Square roots were extracted.

Percentages were used in many situations.

English and metric units were employed. Frequently units were less than 1, as $\frac{1}{100}$ and $\frac{1}{1{,}000}$.

In general, the denominators of the fractions were small and did not exceed 64.

Summary of the arithmetic found in pupils' notebooks.—The findings relating to the arithmetic used by pupils in early secondary-

school subjects other than mathematics show that the striking feature of the arithmetic is its simplicity. As the pupil advances toward the senior high school the operations that he is called upon to perform increase in complexity.

The pupil should master the following facts and processes of arithmetic:

1. Addition, subtraction, multiplication, and division of whole numbers, common and decimal fractions, and mixed numbers. Skill in these processes is absolutely essential. The emphasis, however, is on simple operations. The size of the whole numbers, in general, is not beyond four figures. The number of addends is usually small. The number of factors most common in multiplication problems is 2. Ability to find the square root of a number is required in the solutions of some of the problems.

2. The presence of large numbers demands a control of the number system sufficient to enable the pupil to appreciate, read, and understand large numbers up to billions. Large numbers are frequently compared by means of ratio.

3. Percentages occur in a variety of forms. Ratio and percentages have a prominent place.

4. There is an abundance of experiences in changing from one unit to another, as in changing inches to centimeters, common fractions to decimal fractions, and mixed numbers to fractions.

5. A variety of units of measure is found.

6. The denominators of the most commonly used fractions are small.

The study further suggests that in the computations used in secondary-school mathematics much emphasis should be placed on accuracy. Pupils should become well grounded in the fundamentals, and the secondary school should provide much training along that line. Furthermore, the arithmetical difficulties experienced in various school subjects do not lie so much in the pupil's inability to perform isolated processes, but more in performing them accurately when they occur in new situations. Hence, there should be much correlation of arithmetic with problem situations.

If actual use in school subjects is considered, the amount of computational arithmetic now traditionally taught in Grades VII and VIII is greater than required in the various subjects. Continued

emphasis should be given on the correct use of arithmetic in all mathematical courses from the seventh grade upward.

Geometry used in school subjects.—The notebooks referred to in the foregoing pages supplied in addition to the arithmetical facts much information about the geometry which pupils are expected to

TABLE XVII

GEOMETRIC CONCEPTS USED IN SCHOOL SUBJECTS

General Terms	Line or Lines	Angle	Polygon	Circle	Units	Solids
Area	Curved line	Acute angle	Hexagon	Arc	Acre	Area of solid
Diagram	Diagonal	Central angle	Octagon	Center	Foot	Cone
Dimensions	Distance from line	Degree	Pentagon	Circumference	Inch	Cylinder
Direction	Irregular line	Minute	Rectangle	Concentric circles	Meter	Frustum
Distance	Length of line	Obtuse angle	Square	Diameter	Mile	Prism
Ellipse	Midpoint of line	Right angle	Trapezoid	Intercepted arc	Square	Pyramid
Equidistant	Oblique line	Sides of angle	Triangle	Latitude		Sphere
Graph	Opposite sides	Straight angle		Longitude		Volume of solid
Height	Parallel lines					
Intersect	Perimeter					
Length	Perpendicular lines					
Measuring	Segment					
Midpoint	Sun's rays					
Oval	Straight line					
Point	Vertical line					
Scale						
Shape						
Side						
Size						
Slope						
Surface						
Thickness						
Turn						
Width						

know. They show that the character of the geometry is informational, constructional, mensurational, intuitive, but not demonstrative. Table XVII gives a list of geometric concepts which should be understood by pupils taking seventh-grade courses in sewing, manual training, metal work, and high-school courses in general science, general shop, printing, and geography. The amount of needed geometric information is surprising, and many of the con-

cepts used in the courses are not taught or even mentioned in classes in mathematics below the tenth and eleventh grades.

Table XVIII is a summary of geometric activities disclosed in the notebooks.

It was found that pupils had to be able to use the following instruments: ruler, compasses, dividers, trysquare, protractor, and squared paper. Tables XVII and XVIII show an abundance of

TABLE XVIII

Geometric Activities in School Subjects

Reading	Measuring	Drawing	Making Geometric Constructions
Figures by letters	Estimating lengths and angles	Angle	Adding angles
Interpreting maps and graphs	Measuring line segments	Arc	Adding line segments
	Measuring parts in scale drawings	Circle	Bisecting an angle
		Cone	Constructing an angle of a given size
		Curved line	Constructing parallels
		Cylinder	Constructing perpendiculars
		Design	Dividing a circle into equal parts
		Diagram	Dividing a line segment into equal parts
		Ellipse	Laying off a line segment
		Free-hand drawing	Locating the center of a circle
		Frustum	
		Graph	
		Parallels	
		Perpendiculars	
		Polygon	
		Prism	
		Pyramid	
		Rectangular block	
		Sketch	
		Straight line	
		To scale	

geometric subject matter that is not commonly taught in the mathematics courses of the seventh and eighth grades, but is needed by the pupils in their work in other courses.

Algebra used in school subjects.—The amount of algebra found in the early school subjects other than mathematics is small. The following terms are used: "equal," "term," "collecting terms," "variable," "variation," "proportion," "direct" and "inverse variation," "formula," "equation," and "linear equation." Algebraic notations, especially letters, are used in drawings to denote dimensions. Ability to solve proportions when three of the four terms are given is essential. Formulas are found but they are used almost en-

tirely as rules for making computations. Functional relations are not brought out.

As the pupil progresses toward the ninth grade and the higher grades, the demand for algebraic knowledge increases. Thus in the course in electricity he is expected to understand the solution of linear equations; to state verbal laws in algebraic symbols; to solve proportions; to understand parentheses which indicate addition or subtraction of binomials; and to solve for various letters such formulas as $x=2.54y$, $I=I_1+I_2+I_3$, $I=\frac{1}{R}E$, and $R=\frac{KL}{d^2}$.

Mathematics used in general science.—An analysis of a textbook in first-year science disclosed that 29 per cent of the mathematics involved is concerned with the linear equation and formula; 12 per cent with graphs; 18 per cent with scale drawings, pictures, and diagrams; and 2.5 per cent with angles (see Fig. 17).

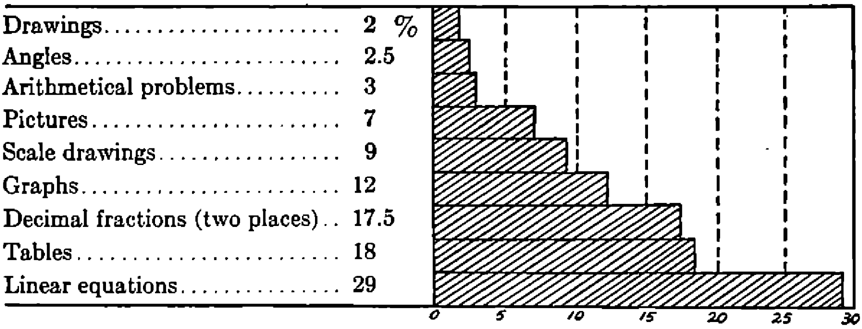


Fig. 17.—Percentages of mathematical content in first-year science

It is a matter of importance that two aspects of the equation are emphasized. On the one hand, it is a statement calling for the value of an unknown number; on the other, it is an expression of relationship between several variables. The following are typical formulas:

$$\text{Current strength} = \frac{\text{Electromotive force}}{\text{Resistance}}$$

$$\text{Amperes} = \frac{\text{Volts}}{\text{Ohms}}$$

$$\text{Work} = \text{Force} \times \text{Distance}$$

Tabular representation of numerical facts is in abundance. The pupil should therefore be taught in mathematics to interpret tables of numerical facts. It was found that 18 per cent of the mathematical work in the general-science course is concerned with tabular representation.

The graph is coming more and more into use in school subjects. Various types of graphs are employed, especially the bar graph, line graph, and circular graph. Graphs are also used to picture such precise laws as $x = 2.54y$ and the laws of direct and inverse proportion. The ability to read and interpret graphs seems to be more important than the ability to make graphs.

Mathematics in senior high school and college subjects.—Studies similar to those described in the foregoing pages have been made for several subjects taught on the college level. Rudman[19] gives an outline of the mathematics needed in a four-year sheet-metal course. Williams[20] made an analysis of a textbook in chemistry by Noyes, and solved the problems to see how much mathematics was actually used by the student. He found the arithmetic at times complicated. The following was typical of the arithmetic:

The problems involved adding integers and decimal fractions, subtracting decimal fractions containing as many as six places, multiplying four-place by six-place numbers and dividing complicated decimal fractions.

One-place, two-place, or three-place decimal fractions were common, but decimal fractions with more places were not unusual. One decimal fraction was found to contain eight places.

Decimal fractions frequently were mixed with common fractions. The denominators were usually less than 10. Eight of 43 denominators were greater than 10 and less than 100.

Nineteen denominate units occurred. The most frequent were: gross, liter, and degree.

Mathematical concepts occurred 1,156 times. The algebra was usually simple. He found algebraic addition, substitution into formulas, ratio and proportion, direct and inverse proportion, equa-

[19] Barnet Rudman, "Related Mathematics in the Cooperative Sheet Metal Course," *School Science and Mathematics*, XXXII (October, 1932), 725–35.

[20] L. W. Williams, "The Mathematics Needed in Freshman Chemistry," *School Science and Mathematics*, XXI (October, 1921), 654–65.

tions of the first degree in one and in two unknowns, and fractional equations. The pupil is expected to be able to state the equation for a given problem. He must understand the meaning and manipulation of formulas, and the principles of proportions. Evaluation of formulas calls for understanding of algebraic symbols and proficiency in arithmetical computation.

Rendahl[21] made a study of high-school chemistry to discover the mathematics in that course needed to solve problems. A list of 396 problems was selected from three textbooks. Each problem was then solved by the method most commonly used by pupils. The following conclusions were drawn from the findings:

1. The fundamental operations on whole numbers and decimals are slightly complicated by large numbers and numerous decimals.
2. Multiplication is the only operation used upon common fractions. Fractions with large numerators and denominators further complicated by decimals are found in the problems of some texts.
3. The three variations of the percentage problem are found in all of the texts examined.
4. It is possible to avoid the use of proportion in solving chemistry problems.
5. The more important uses of algebra occur in solving simple and fractional equations of the first degree and with one unknown, and in substituting in a formula.
6. The use of geometry is negligible.
7. Only 34 different units of measure are used in the three books. Thirty-eight per cent of the units are based on the metric system.
8. Not over 54 different mathematical terms are found in the vocabulary of the problems from any one book. Sixty-two per cent of them are among the 10,000 commonest words.
9. There is considerable variation in the texts examined as to the number of mathematical problems, the use of large fractions, the use of proportion, fractional equations, and the metric system.

Neville[22] reports that his experience has convinced him that success in chemistry is related to the ability to analyze simple problems in arithmetic, "to reason arithmetically." The type of prob-

[21] J. L. Rendahl, "The Mathematics Used in Solving Problems in High School Chemistry," *ibid.*, XXX (June, 1930), 683–89.

[22] Harvey A. Neville, "Mathematics and Science," *Mathematics Teacher*, XX (January, 1927), 19–25.

lems pupils should be able to solve is illustrated in the following test:

1. 16 is what per cent of 25?
2. Solve for x: $\frac{16.4}{4.8}=\frac{5.5}{x}$.
3. If a copper ore contains only 3 per cent copper, how many tons of the ore must be worked in order to obtain a ton of pure copper?
4. If Champaign is 120 miles from Chicago, what is the distance in kilometers? (A kilometer$=\frac{5}{8}$ mile.)
5. If 27 pints of grape juice cost $5.40, what is the price of grape juice per gallon?
6. If 400 gallons of water flow through a pipe in 9 hours, how much water flows through in 4 days?
7. There are in a room 30 women, 18 men, 15 children. What per cent of the group are men?
8. In a certain chemistry class 68 per cent passed; if 48 students failed, what was the total number of students in the class?

A committee from the departments of mathematics, physics, and chemistry of the University of Pennsylvania[23] has compiled a list of processes necessary for successful study of beginning courses in physics and chemistry. Students expecting to take Chemistry I are given a list of forty-one type problems and are advised either to do outside work or to postpone the study of the beginning courses until they possess a ready working knowledge of the problems. The following understandings and processes are involved in the problems: operations with decimal fractions, relations between decimal and common fractions, relations between decimal fractions and percentage, operations with signed numbers, logarithmic computation, positive and negative exponents, roots, solution of linear and fractional equations, and complex fractions of a simple type.

A second list of problems, intended as a preparation for physics, involves a knowledge of the relations between common and decimal fractions, complex fractions, exponents, solution of simple simultaneous and quadratic equations, trigonometric functions, the fundamental trigonometric identities, and simple verbal problems.

Teachers of eleventh- and twelfth-grade mathematics will find in

[23] H. B. Evans, "Cooperation in Mathematics and Science," *ibid.*, XXV (January, 1932), 17–21.

the lists a considerable amount of material which they should emphasize in their courses. Not only will they be aided in preparing the pupils for the mathematical needs of physics and chemistry, but they will find all of the subject matter profitable from the point of view of mathematics. The importance of emphasis on arithmetic in all courses of secondary mathematics is apparent.

Pressey[24] made an analysis of textbooks in mathematics, physics, and chemistry by working all the problems and counting the frequencies with which different concepts and operations occurred. Her purpose was to determine the mathematical needs in the Freshman courses in the three subjects. The results are classified in four divisions: units of measure, formulas, arithmetic and algebraic operations, and mathematical concepts. Twenty-nine units appeared five times or more. She found 172 different formulas, which the students should be able to read, understand, and evaluate; gave a list of seventeen arithmetic and eleven algebraic skills and their frequencies; and named twenty-eight concepts which seem essential for Freshman work in the three subjects. There is nothing in the findings that is not taught by most teachers of high-school mathematics. However, a knowledge of the subject matter disclosed in the study will enable teachers of mathematics to give it the deserved emphasis.

Kilzer[25] analyzed five textbooks in physics and then made an inventory test covering the mathematics needed for the course. The test is to be given during the first week of the physics course. Teachers of mathematics may emphasize without loss to the pupils the types of mathematics which the test contains. It was found that most of the mathematics needed in solving problems in physics is not at all difficult. Very little trigonometry is used. The test has been published by the Public School Publishing Company, Bloomington, Illinois.

Zerbe[26] examined ten geometry textbooks to determine the

[24] Luella C. Pressey, "The Needs of Freshmen in the Field of Mathematics," *School Science and Mathematics*, XXX (March, 1930), 238–43.

[25] L. R. Kilzer, "The Mathematics Needed in High School Physics," *ibid.*, XXIX (April, 1929), 360–62.

[26] Hobson M. Zerbe, "The Elements of Plane Geometry in High School Physics," *ibid.*, XXX (June, 1930), 665–67.

theorems, constructions, and terms which students should know when they study high-school physics. He collected a list of twenty-five theorems and ten constructions.

On the basis of an analysis of forty texts Thorndike and Woodyard reached the conclusions stated on page 174.

An investigation to determine the mathematics used in a college Freshman course in clothing was made by Gallaway,[27] who found the following:

The integers contained mostly less than four places.

Roman numerals were frequent.

The units of measure were inch, dollar, yard, head, and year.

Fractions contained the denominators 2, 4, 6, 8, 10, 12, 16.

Few cases of decimal fractions were found.

The use of "per cent" was frequent.

The operations were: adding fractions to whole or mixed numbers, subtracting whole or mixed numbers from whole or mixed numbers; multiplying whole or mixed numbers by fractions; finding a fraction of a fraction; reducing mixed numbers to fractions and fractions to mixed numbers; reducing yards to inches and inches to yards.

The integers mostly contained less than four places.

Roman numerals were frequent.

Rhetorical equations had to be translated into symbolic equations.

Out of 111 mathematical concepts found in the text, 43 were geometric, the most common being: line, point, center, width, length, measure, equal.

In the foregoing pages the method of textbook analysis has been used to aid in the choice of subject matter needed as a foundation for future courses in mathematics and for other courses in which a knowledge of mathematics is essential. The use of such subject matter should enrich the mathematical courses because it brings to them interesting applications drawn from other fields.

Further materials may be found in college entrance examinations, committee reports, in mathematical magazines and yearbooks, and in the bibliography at the end of this chapter. College entrance examinations have always influenced the authors of text-

[27] W. W. Charters, *Curriculum Construction*, pp. 241–43.

books. They have been severely criticized because they include a type of subject matter which seems to serve no other purpose than that of preparing pupils to pass examinations. On the other hand, they may become powerful instruments of progress if they support the recommendations and constructive suggestions of educational leaders and authoritative committees. For example, the list of minimum essentials made up by the College Entrance Examination Board is helpful to the teachers who find it impossible to do all the work contained in the textbook and therefore are unable to introduce supplementary materials in which they are interested.

Mathematics needed in life-activities.—In selecting instructional materials the teacher should take into consideration the needs of people in their homes, in their daily work, in business, and in industry. Numerous studies have been made to determine this socially valuable material. They have disclosed much information that may be used to adapt mathematical instruction to the needs of the community and to aid pupils in understanding the world they live in by teaching them the facts and principles of mathematics needed in the activities of everyday life. Some investigators have erroneously interpreted the findings of these studies to mean that the materials found to have a definite need in the life of the pupil are the only materials to be used in instruction. It should not be concluded that the mathematics actually used by people are necessarily identical with that which should or should not be used.

> The curriculum maker who thinks that he has exhausted the catalogue of uses of number when he has listed the examples which ordinary men solve in a day or week is superficial to such an extreme degree that he is an unsafe guide in arranging the plans of the school.[28]

The techniques employed in the determination of the materials of social value are:

1. The analysis of the mathematical interest of pupils based upon the opinions of pupils and of adults who are competent to judge.

2. The tabulation of the judgments of teachers qualified to speak with authority as to the mathematical needs of the average citizen.

[28] Charles H. Judd, "The Fallacy of Treating School Subjects as Tool Subjects," *Third Yearbook: National Council of Teachers of Mathematics* (1928), pp. 3, 4.

3. The tabulation and analysis of mathematics used in pupil activities, in adult activities, in literature of a general nature, and in reading of a technical nature.

Mathematical needs and interests of pupils may be determined by observing them, by examining their activities in detail, and by collecting information from questionnaires filled out by pupils and teachers. Before the material thus collected is used as subject matter for the courses in mathematics it should be subjected to careful examination. The questionnaire method has several limitations. The questions are not always stated clearly and are therefore misunderstood, and the persons answering the questionnaires do not always give them careful attention.

The studies of the mathematics needed in the home disclose many opportunities for relating school work in mathematics to the home. Bills from the grocer, dry-goods store, and gas and electric companies; personal accounts and housekeeping budgets; buying and selling of goods; saving and borrowing money; rent, investment, insurance, and taxes; and many other matters of a quantitative nature fall within the pupils' experiences in his home life. There is an abundance of material that may be used to advantage in the study of mathematics at all levels of the secondary school.

Arithmetic in adult life-activities.—In the following pages are summarized the results of several investigations which aim to discover the social usage of mathematics, to find out what mathematics the average well-informed adult must know to carry on successfully his everyday life-activities, and to establish the amount and types of mathematics used by people as they go about their everyday tasks. From the mass of materials collected by such studies, that which is of immediate value to the learner should be selected for teaching purposes. Subject matter that has value only as a preparation for future courses and later life-needs of the pupil is secondary as a basis for the curriculum and should be omitted or deferred to later courses.

Table XIX summarizes the findings of several studies concerned with the size of numbers used by adults in problems and reading periodicals and newspapers. It is seen that few problems which adults solve involve more than five-figure numbers. In reading, however, the results are radically different. Some very large num-

TABLE XIX

Sizes of Numbers Used by Adults

Name of Investigator	Operation	Type of Investigation	Number of Places
Adams, H. W.....	In general reading	The mathematics encountered in general reading of newspapers and periodicals	Numbers varied from two to twelve places; most of them were two-place numbers.
Charters, W. W...	Addition in problems	Department-store arithmetic	90 per cent had four places or less. The maximum was five places. Few problems had more than three addends. Salespeople use the additive method of subtraction.
Charters, W. W...	Multiplication in problems	Department-store arithmetic	In 97 per cent of cases the multiplier was twelve or less. The multiplicand was three places or less.
Hansen, Einar A...	Addition in problems	The arithmetic of a salesperson's tally cards	10 per cent had five places, 73 per cent had four places, 17 per cent had three places.
Wilson, Guy M....	Addition in problems	A survey of the social and business uses of arithmetic	97 per cent had four places or less.
Wilson, Guy M....	Division in problems	A survey of the social and business uses of arithmetic	39.6 per cent of all problems had four-place numbers in the divisor and 43.4 per cent had two places in the divisor.
Wilson, Guy M....	Multiplication in problems	A survey of the social and business use of arithmetic	Almost all had one- or two-place numbers in the multiplier.
Woody, Clifford...	Addition in problems	Types of arithmetic needed in certain types of salesmanship	Not over five places in addends, commonly three places.

TABLE XIX—*Continued*

Name of Investigator	Operation	Type of Investigation	Number of Places
Woody, Clifford...	Division in problems	Types of arithmetic needed in certain types of salesmanship	Commonly two or three places were found in either divisor or dividend.
Woody, Clifford...	Multiplication in problems	Types of arithmetic needed in certain types of salesmanship	Two or three places in the multiplicand and one or two places in the multiplier.
Woody, Clifford...	Subtraction in problems	Types of arithmetic needed in certain types of salesmanship	Unusually three or four figures in the minuend.

bers are encountered. It follows, therefore, that in teaching mathematics small numbers might have preference in calculations, and that a sufficient number of situations should be provided which involve large numbers to enable the pupil to acquire an understanding and control of the number system.

Table XX summarizes the findings as to the size of fractions that are employed in social use. The various investigations are in striking agreement as to the types of fractions which adults meet in the problems of everyday life. The denominators are small, seldom exceeding 64. Emphasis in teaching the operations with fractions may therefore be on the simple fractions.

Table XXI shows but limited social use of decimal fractions, except when they occur in connection with United States money. The investigation of Adams, however, shows a wide occurrence of decimal fractions in newspapers and periodicals. There is a surprising lack of use of per cents and percentage. Discount rates in advertisements are frequently stated in per cent.

Table XXII is a summary of the findings relative to the operations in which manipulative skill is essential. It shows that the operations which adults have to perform are few and are far more simple than the corresponding work offered in textbooks on arithmetic.

Table XXIII contains a list of topics which possess social value. They are entitled to attention more because they have informational value than because they offer training in arithmetical computation. The asterisks in Table XXIII indicate the investigators who found the topics socially useful.

TABLE XX

TYPES OF FRACTIONS FOUND IN SOCIAL AND BUSINESS USE

Name of Investigator	Type of Investigation	Size of Denominators
Adams, H. W.	The mathematics encountered in general readings of newpapers and periodicals	Only 6 out of 3,000 fractions have denominators larger than 16.
Charters, W. W.	Department-store arithmetic	81 per cent of the denominators are 2 and 4, 6 per cent are 16, and 3 per cent are 8. The largest is 16. The denominators of 99 per cent of the fractions are 2, 3, 4, 5, 6, 8, 10, 12, 16.
Mitchell, H. Edwin	Social demands on the course of study in arithmetic:	
	In factory pay-rolls	Denominators emphasized are: 3, 4, 6, 12. The word "dozen" and fractional parts of 12 are used very much. Addition and multiplication of fractions.
	In the hardware store catalogue	Denominators used are 16, 32, 64. Mixed numbers with denominator 12.
Wilson, Guy M.	A survey of the social and business uses of arithmetic	Most commonly used denominators are: 2, 3, 4, 5, 8.
Wise, Carl T.	Survey of arithmetical problems	93.9 per cent of all fractions had denominators: 2, 3, 4, 5, 8.
Woody, Clifford	Types of arithmetic needed in certain types of salesmanship	The most commonly used denominators are: 2, 4, 6, 12.

TABLE XXI

SOCIAL USE OF DECIMAL FRACTIONS AND PERCENTAGES

Name of Investigator	Use of Decimal Fractions	Use of Percentages
Adams, H. W......	Decimal fractions occur frequently, mostly having less than four places, ranging from .000012 to .97	A variety of percentages: $1\frac{1}{2}$, $1\frac{3}{4}$, 2, $2\frac{3}{4}$, 3, $33\frac{1}{3}$, $99\frac{44}{100}$, 177, 700, 1,000, 1,300.
Charters, W. W....	Very limited use of decimals, occurring only in 10, 15, and 20 per cent	
Mitchell, H. Edwin	Addition of decimals in terms of dollars and cents	Discount rates are expressed in percentages, 10, 20, 25 per cent being most commonly used
Wilson, Guy M.	Little use made of decimals	Meager support for cases (some problems in percentage are found)
Woody, Clifford...	No use of decimals except connection with United States money	Discount rates: 5, 10, 20, 25, 30, 35, 40, 50

TABLE XXII

ARITHMETICAL OPERATIONS COMMONLY PERFORMED BY ADULTS

Name of Investigator	Operations
Adams, H. W........	No problems of the textbook type.
Charters, W. W......	Few problems had more than 3 addends. The maximum number of addends was 18.
Mitchell, H. Edwin...	Addition of all possible fractions with 12 as denominator to integers and mixed numbers, of decimals in terms of dollars and cents. Adding United States money. Multiplication by fractional, integral, or mixed numbers.
Wilson, Guy M......	90.6 per cent of all problems involve addition, subtraction, multiplication, division, and fractions.
Woody, Clifford......	Most common are addition of two- or three-place numbers. Few problems in subtraction. Multiplication of two- or three-place numbers by one- or two-place numbers. Division problems usually have two- or three-place numbers in dividend and divisor. Multiplication of simple fractions by three- or four-place numbers or by mixed numbers having two or three places.

Summary of the social uses of arithmetic.—The preceding studies show that in considering the social value of arithmetic as a basis for selecting suitable material for the secondary-school course in mathematics, teachers may keep in mind that the amount of arithmetic used by adults is surprisingly small. Owing to the fact that much of the material which is useful to adults has little meaning to

TABLE XXIII

Socially Useful Topics of Arithmetic

Topics	Names of Investigators							
	Wilson	Wise	Jessup-Coffman	Adams	Moore	Mitchell	Camerer	Woody
Insurance	*	*	*	.....	.....	.....	.....	.....
Taxation	.....	.....	*	.....	.....	.....	.....	.....
Public expenditures	.....	.....	*	.....	.....	.....	.....	.....
Levies	.....	.....	*	.....	.....	.....	.....	.....
Banking	*	.....	*	.....	*	.....	*	.....
Interest	*	*	.....	.....	*	.....	*	.....
Saving and loaning money	*	.....	*	.....	.....	.....	.....	.....
Keeping simple accounts	*	.....	.....	.....	*	.....	.....	.....
Investments	*	.....	.....	.....	.....	.....	.....	.....
Mortgages	*	.....	.....	.....	.....	.....	.....	.....
Building and Loan associations	*	.....	*	.....	.....	.....	.....	.....
Bonds	*	.....	*	.....	.....	.....	.....	.....
Discount	.....	*	.....	.....	*	*	.....	*
Stocks	.....	.....	*	.....	.....	.....	.....	.....
Profits	.....	.....	*	.....	.....	.....	.....	.....
Rent	*	.....	.....	.....	.....	.....	.....	.....
Buying, selling	*	.....	.....	*	.....	.....	.....	.....
Weight and measures	.....	*	.....	*	.....	.....	.....	.....
Labor, wages	*	.....	.....	.....	.....	.....	.....	.....
Painting, masonry	.....	*	.....	.....	.....	.....	.....	.....

the pupil because it falls outside of his experiences, the actual amount to be offered is further limited. This suggests that much of the material commonly taught in Grades VII and VIII might be deferred to the time when the pupil has reached a stage of maturity at which his experiences enable him to grasp the meaning of this social-economic subject matter.

On the basis of the findings of the investigations the following recommendations may be made regarding the choice of arithmetical

materials that deserve special emphasis in courses in secondary-school mathematics:

1. Problems and computations should make use of numbers containing from one to five figures. The pupil should also have experience with large numbers sufficient to attain an appreciation of the number system, i.e., he should learn to read and understand numbers with more than five figures.
2. The majority of fractions should have small denominators, such as 64 or less.
3. Decimal fractions used in the fundamental operations should contain four places or less. Attention should be given to transactions involving decimal fractions.
4. Percentage should be thoroughly taught. Emphasis should be given to the applications of percentage, e.g., to discount.
5. Practice should be provided in the fundamental operations with integers and fractions. Much problem work should be offered which involves these operations. The majority of computations aiming to develop manipulative skill and accuracy should be of a simple type.
6. Preference should be given to topics which are known to be socially useful. Some topics that are not socially useful and are still retained in textbooks should receive less emphasis or may be eliminated. Since the number of problems involving taxes, stocks, and bonds is surprisingly small, less use than is customary at present might be made of these topics as a means of training in arithmetical computation. The meanings of these terms should be taught because of their informational value.
7. The problems of adults are unlike the typical textbook problems. Emphasis should be on problems involving selling and buying.

Algebra and geometry used by adults.—Most of the investigations reported in the foregoing pages were made for the purpose of determining the social uses of arithmetic. However, some of them discovered references to algebraic and geometric material. Thus Adams[29] mentions that graphs are encountered in reading of newspapers and magazines, and that he found some terms having geo-

[29] H. W. Adams, *Uses of Mathematics in General Reading* (Master's thesis, Department of Education, University of Chicago, 1924).

metric meaning, as $40 \times 18 \times 16$. However, he reports no further uses of algebra and geometry. Touton[30] reports forty-two mathematical concepts per page in three issues of the *American Magazine.* Among them were listed simple equations, graphs, formulas, and signed numbers. Thorndike and Woodyard[31] examined the *Encyclopaedia Britannica,* noting all references to mathematics beyond arithmetic. The first two hundred pages of each volume, from I to XXVIII, were read and the mathematical references collected. They report that 3.57 per cent of the articles used mathematics beyond arithmetic. Table XXIV illustrates the character of the findings.

TABLE XXIV

ENCYCLOPAEDIA COUNTS

	No. of Articles Concerned	No. Linear-Inch Space Utilized
Mathematical definitions	12	22
Long articles with slight mathematics	7	3,714
Requiring vocabulary of geometric shapes:		
Description	39	874
Shapes of buildings, lands, etc.	60	2,301
Other uses of terms	68	4,606
Biographies of mathematicians	32	663
Requiring algebra only:		
Graph	5	375
Formula	2	123
Requiring more than elementary algebra:	44	7,063
Total mathematics usage	269	19,741
Total examined	7,551	106,400

A large variety of geometric terms was used which require understanding of forms of two and three dimensions. It appears that in numerous fields of study ability to understand formulas and graphs is highly essential. Writers who lack this ability are compelled to use unnecessarily long and clumsy descriptions. Schorling examined issues of *Popular Mechanics* and *Popular Science* and found 211 terms of geometric nature varying in frequency from 1 to 507.

[30] Frank C. Touton, "Mathematical Concepts in Current Literature," *School Science and Mathematics*, XXIII (October, 1923), 648–55.

[31] Edward L. Thorndike, *The Psychology of Algebra*, p. 83.

From an investigation aiming to determine the amount of geometry used by adults Chase[32] constructs Table XXV, in which she gives the percentages of men and women engaged in various occupations in which they use certain geometric concepts in their life out of school.

The table shows that the percentage of people using these fundamental geometric concepts is small. However, the investigation included only one hundred and forty men and women and can hardly be considered representative.

Charters reports an analysis of the geometry required in a machinist's job of turning a cylinder, in which the following geo-

TABLE XXV

PERCENTAGE OF ADULTS USING CERTAIN GEOMETRIC CONCEPTS

	Area Of					Area Of				Volume Of		
	Sq.	Rec.	Tri.	Cir.	Rhom.	Cube	Pri.	Cyl.	Pyr.	Cube	Pri.	Cyl.
Percentage of Men	$66\frac{2}{3}$	61	42	32	11	28	20	24	9	39	18	39
Percentage of Women	27	26	8	12	3	8	5	5	0	9	8	9

metric terms must be understood: "straight line," "parallel lines," "perpendicular lines," "vertical lines," "angle," "angle bisector," "circle," "radius," "diameter," "concentric circles," "cylinder," and elements of "cylinder."

Mathematics in industries.—Surveys of the mathematical needs of industries find many problems that may be included among the problems to be solved by pupils at various school levels. Care must be taken not to choose problems that are too remote from the pupil's experiences to be understood and to exclude materials that are too complicated. Morgan[33] made a study of trade-school texts and corporation-school material. She concluded that it requires

[32] Sara E. Chase, "Waste in Arithmetic," *Teachers College Record,* XVIII (September, 1917), 350–71.

[33] Florence J. Morgan, *The Use of Secondary Mathematics in Certain Industrial Occupations* (unpublished Master's thesis, Department of Education, University of Chicago, 1918), p. 86.

about eighteen weeks to teach the algebra essential in the field of industrial occupation. Harold[34] classified the problems found in seven correspondence school texts, five popular handbooks, and two issues of three non-technical magazines to determine the amount and type of mathematics needed in the general field of automobile engineering. He reported 37,439 different items, including 101 algebra problems which he classified as follows:

Fractional linear equations	66	Non-fractional quadratic equations	5
Fractional quadratic equations	17	Addition	2
Non-fractional linear equations	11		

Leonard[35] collected a list of problems from men taking courses in an evening school. He was led to the following conclusions:

1. Students going into technical work need all the mathematics ordinarily given for college entrance and more in addition.
2. Students not going to college should have a general mathematical training, no one line of work seems to confine itself to one line of mathematics.
3. Some of the weaker students going through high school are not mathematically prepared for success in the industries.
4. The handbook does not remove the necessities of learning the principles of mathematics.
5. The two things most commonly met are the formula and the triangle. The formula requires all the devices of algebra and logarithms, and the triangle calls for a knowledge of all the facts of geometry and trigonometry.
6. Students who cannot do arithmetic with facility cannot do problems enough of this type to get sufficient drill.
7. To the man in the shop a good knowledge of arithmetic is worth far more than a meager knowledge of arithmetic and algebra.
8. Men in the industry who cannot do arithmetic, simple algebra, and trigonometry readily do not have mathematics enough to do them much good.
9. Mathematics is necessary as a matter of preparedness for problems that are bound to spring up without warning.

[34] Russel H. Harold, *A Study of the Mathematics Involved in the Field of Auto Mechanics* (unpublished Master's thesis, Department of Education, University of Chicago, 1925), p. 116.

[35] C. J. Leonard, "Mathematics in Industry," *School Science and Mathematics*, XXIX (March, 1929), 245–55.

Supervisors working on the problem of selecting instructional materials will find it profitable to consult with persons engaged in local industries and to analyze the problems which men working in the industries have to solve.

BIBLIOGRAPHY

Adams, H. W. *Uses of Mathematics in General Reading.* Master's thesis, Department of Education, University of Chicago, 1924.

Betz, William. "Whither Algebra? A Challenge and a Plea," *Mathematics Teacher*, XXIII (February, 1930), 1905–25.

Bobbitt, Franklin K. *Curriculum Investigations*, "Supplementary Educational Monographs," No. 31. Department of Education, University of Chicago, 1926.

———. *Curriculum Making in Los Angeles*, "Supplementary Educational Monographs," No. 20, Department of Education, University of Chicago, 1922.

———. "The Mathematics Used in Popular Science," *Curriculum Investigation*, "Supplementary Educational Monographs," No. 31, Department of Education, University of Chicago, 1926.

———. "The New Technique of Curriculum Making in Arithmetic," *Elementary School Journal*, XXV (September, October, 1924), 45–54, 127–43.

Bobo, H. R. *Analysis of the Mathematics Found in Books on Popular Science.* Master's thesis, University of Chicago, 1925.

Bowers, W. G. "Mathematical Problems in Elementary Chemistry," *School Science and Mathematics*, XXVIII (December, 1928), 975–80.

Camerer, Alice. "What Shall Be the Minimum Information about Banking?" *Seventh Yearbook: National Society for the Study of Education*, Part I (1918).

Clarke, Edith. "Mathematics in Modern Business," *Mathematics Teacher*, XXI (May, 1928), 259–67.

Colwell, Lewis W. "Arithmetic in the Junior High School," *School Science and Mathematics*, XXV (February, 1925), 171–78.

Congdon, Allan R. *Training in High School Mathematics Essential for Success in Certain College Subjects*, "Teachers College Contributions to Education," No. 403. New York: Bureau of Publications, Teachers College, Columbia University, 1930.

Cragg, Maud Elizabeth. *The Mathematics Found in Social Science Textbooks*, Master's thesis, Department of Education, University of Chicago, 1925.

Edwards, William H. "Trigonometric Formulae Encountered in a College Engineering Course," *School Science and Mathematics*, XXVIII (March, 1928), 239–43.

Evans, H. B. "Co-operation in Mathematics and Science," *Mathematics Teacher*, XXV (January, 1932), 17–21.

Fernelius, W. Conard. "The Applications of Mathematics to Chemistry," *School Science and Mathematics*, XXIX (January, 1929), 71–78.

Franklin, G. T. "Some Simple Uses of Mathematics To Clarify Chemical Principles," *ibid.*, pp. 494–96.

Hill, George E. "The Vocational Uses of Elementary High School Algebra," *ibid.*, XXXII (June, 1932), 641–43.

Johnson, J. T. "Geometry in the Junior High School," *ibid.*, XXV (June, 1925), 611–17.

Jones, Gertrude. "A Few Determinants in Building the Course of Study in Mathematics," *Mathematics Teacher*, XXII (November, 1929), 397–404.

Kilzer, L. R. "The Mathematics Needed in High School Physics," *School Science and Mathematics*, XXIX (April, 1929), 360–62.

Lemmer, Jerome G. "Is High School Mathematics an Adequate Preparation for High School Physics?" *ibid.*, XXX (January, 1930), 41–44.

Leonard, C. J. "Mathematics in Industry," *ibid.*, XXIX (March, 1929), 245–55.

McLear, Martha. "Mathematics in Current Literature," *Pedagogical Seminary*, XXX (March, 1923), 48–50.

Mitchell, H. Edwin. "Some Social Demands of the Course of Study in Arithmetic," *Seventh Yearbook: National Society for the Study of Education*, Part I (1918).

Moore, Charles N. "The Future Development of Mathematical Education," *Mathematics Teacher*, XV (December, 1922), 478–82.

———. "Mathematics and the Future," *ibid.*, XXII (April, 1929), 203–14.

National Council of Teachers of Mathematics. *Sixth Yearbook: Mathematics in Modern Life*. New York: Teachers College, Columbia University, 1931.

Neville, Harvey A. "Mathematics and Science," *Mathematics Teacher*, XX (January, 1927), 19–25.

Nyberg, Joseph A. "A Discussion of an Article on Mathematical Abilities and Physics," *School Science and Mathematics*, XXVI (January, 1926), 9–15.

———. "Recent Changes in the Teaching of Algebra," *Mathematics Teacher*, XVIII (January, 1925), 10–21.

Perry, Winona. "Has Algebra Certain Real Values for the High School Student of Today?" *ibid.*, XX (November, 1927), 403–6.

Prendergast, Ella M. "Mathematics and Social Science," *ibid.*, XXVI (January, 1933), 33–39.

Pressey, Luella C. "The Needs of Freshmen in the Field of Mathematics," *School Science and Mathematics*, XXX (March, 1930), 238–43.

Reagan, G. W. "The Mathematics Involved in Solving High School Physics Problems," *ibid.*, XXV (March, 1925), 292–99.

Rendahl, J. L. "The Mathematics Used in Solving Problems in High School Chemistry," *ibid.*, XXX (June, 1930), 683–89.

Rich, L. Ashley. "An Argument for a Correlated Course in Science and Algebra," *Mathematics Teacher*, XXV (January, 1932), 33–35.

Rudman, Barnet. "Related Mathematics in the Co-operative Sheet Metal Course," *School Science and Mathematics*, XXXII (October, 1932), 725–35.

Schorling, Raleigh, and Clark, John R. "A Program of Investigation and Cooperative Experimentation in the Mathematics of the Seventh, Eighth, and Ninth School Years," *Mathematics Teacher*, XIV (May, 1921), 264–75.

Schreiber, Edwin. "Some Ideals in Teaching Mathematics," *ibid.*, pp. 252–55.

Simons, Leo G. *Introduction of Algebra into American Schools in the Eighteenth Century*. Washington, 1924.

Straley, H. W. "The Deficiency in Mathematical Training Required of Geologists," *School Science and Mathematics*, XXXII (October, 1932), 745–77.

Thorndike, Edward L. *The Psychology of Algebra*. New York: Macmillan Co., 1923.

Touton, Frank C. "Mathematical Concepts in Current Literature," *School Science and Mathematics*, XXIII (October, 1923), 648–55.

Tyler, H. W. "Mathematics in Science," *Mathematics Teacher*, XXI (May, 1928), 273–79.

Wiener, William. "The Place of Arithmetic in the High School Curriculum," *ibid.*, X (June, 1918), 175–78.

Williams, L. D. "The Mathematics Needed in Freshman Chemistry," *School Science and Mathematics*, XXI (October, 1921), 654–55.

———. "The Application of Scientific Method to the Determination of the Curriculum in Arithmetic," *Journal of Education*, XLI (April 1, 8, 15, 22, 1920), 376–77, 402–3, 431–33, 458–60.

Wilson, G. M. "A Survey of the Social and Business Use of Arithmetic," *Sixteenth Yearbook of the National Society for the Study of Education*, Part I (1917), pp. 128–43.

Wise, Carl T. "A Survey of Arithmetical Problems Arising in Various Occupations," *Elementary School Journal*, XX (October, 1919), 118–36.

Woody, C. "Types of Arithmetic Needed in Certain Types of Salesmanship," *ibid.*, XXII (March, 1922), 505–20.

Young, J. R. "Recent Tendencies in the Teaching of Elementary Applied Mathematics," *School Science and Mathematics*, XVII (March, 1917), 237–44.

Zerbe, Hobson M. "The Elements of Plane Geometry in High School Physics," *ibid.*, XXX (June, 1930), 665–67.

CHAPTER VII

ORGANIZATION OF THE INSTRUCTIONAL MATERIALS OF GEOMETRY

Principles of organization.—The foregoing pages have given some consideration to the question of selecting suitable materials for the study of geometry. Another question is that of arranging the selected materials for teaching purposes. Writers on the subject commonly recommend one or more of the following principles of arrangement of geometric materials:

1. The facts should be presented in the order in which experience and experiment have shown them most easily acquired and mastered by the pupils.
2. As far as possible or advisable the simplest facts should be presented before the more difficult facts are taken up.
3. The arrangement should take into consideration the order in which the race has developed the subject.
4. The first facts should include those which are most commonly used in everyday life and which fall within the experiences of the pupils, especially those needed by the pupils themselves.
5. The study of geometry should proceed gradually and slowly as is the case with arithmetic. If necessary, years may be taken to give the pupil the training necessary to prepare him properly for the study of logical geometry.
6. The subject matter should be arranged in pedagogical units.
7. Geometry should be correlated with the other mathematical subjects and with the natural sciences.

Geometry as a high-school subject has been criticized on the ground that it is merely a course in logic into which the pupil is plunged without purposeful preparation and clear understanding of the basic concepts. The foregoing principles, if carefully applied, will go far toward freeing the teaching of geometry of its most objectionable defects by presenting it in a more teachable form and in psychological as well as in logical sequence.

Geometry acquired before the secondary-school period.—The pupil has had an abundance of geometric experiences long before he enters the seventh grade. He has learned to estimate lengths approximately and to measure with the yardstick. In his play and in other activities he has come to know the meaning of such terms as "line," "angle," "triangle," "rectangle," "square," "circle," "cylinder," and "sphere." Usually he knows something about units of length, surface, and volume.

To be sure, his ideas are hazy and his knowledge is fragmentary. The next step, therefore, will be to determine what he actually knows, to systematize this knowledge, and to round out and extend the geometry he has acquired in previous experiences. This is the beginning of the geometry of the secondary school. Contrary to the traditional order, instruction in geometry should precede the study of algebra. Such an arrangement is psychological and conforms to the way the race has developed mathematics.

The early development of elementary geometry.—Writers on the history of mathematics tell us that geometry had its beginning in simple drawings and designs used to decorate pottery and fabrics.[1] Nearly all the early peoples developed simple geometric forms. Crude designs are in evidence in the early monuments of Mexico, China, India, and Egypt, where they were used for purposes of ornamentation. For the same reason they were used on clothing, rugs, and draperies. In the beginning the designs consisted of simple roughly drawn lines, especially parallel lines. In time the lines were drawn with more care. Soon quadrilaterals, circles, and other plane figures made their appearance. Progress was gradual and improvement steady. Egyptian and Greek civilizations both passed by steps to the more advanced forms until the highest stage of development was reached in the rich gold work of Egypt and Cyprus.[2]

The early stage in the development of geometry suggests that in the beginning instruction in geometry might be concerned with easy drawings, thereby making the pupil acquainted with the simple geometric figures as they occur in decoration and designs.

[1] J. Tropfke, *Geschichte der Elementar Mathematik* (Leipzig, 1922), Vols. I and II.

[2] D. E. Smith, *History of Mathematics* (Boston, 1923), Vol. I.

Indeed, experience has shown that children of the early adolescent period take great delight in making drawings of designs which involve triangles, parallelograms, circles, and others of the common plane figures.[3] Incidentally, as they learn to work with ruler and compasses they may discover from observation many properties of these figures and thus assimilate gradually a definite body of geometric information.

Although much of the geometric content to be selected for the beginning of instruction should be of the aesthetic type, it is not the only type of geometry which is of interest to pupils. They are equally interested in the practical phases of geometry.[4] History shows that the Egyptians and Babylonians developed the practical side of geometry. As early as 4000 B.C. the Egyptians had acquired much geometric information which enabled them to design and build temples and pyramids. In 1850 B.C. they are known to have carried on extensive projects of irrigation and engineering involving surveying, leveling, and mensuration. Herodotus reports that during the reign of Ramesis II (1347 B.C.) the Egyptians were able to survey land for the purpose of dividing it among the people and of determining the rent to be demanded from the owner. Thus, the geometry arose largely from practical necessities. Geometric facts were discovered by methods of observation and verified by experimentation.

The acquired knowledge was carefully preserved by the priests and was gradually passed on to scholars from other countries. The Ahmes papyrus (*ca.* 1650 B.C.) contained formulas for finding the areas of rectangles, trapezoids, triangles, and circles. The Egyptians used these formulas to perform computations of areas of simple surfaces, such as the isosceles triangle and trapezoid. They divided large irregular surfaces into smaller surfaces to make the computation easy. They knew the theorem of Pythagoras long before his time. They understood principles of symmetry, as may be seen in the mural decorations during the period following the building of the pyramids. They had knowledge of the fact that the largest angle of a triangle is a right angle if the sides are in the ratio 3:4:5, a principle which they used in constructing right angles.

[3] E. R. Breslich, *Seventh-Year Mathematics*, chap. iv.

[4] *Ibid.*, chap. iii.

They knew something about geometric solids, e.g., the pyramid. However, since they were not interested in the purely theoretical aspects of geometry, they were unable to advance very far in the development of the subject.

The first stage of instruction in geometry.—The history of early geometry suggests that the work of the first stage of instruction be of the aesthetic and practical types, that as far as possible it should relate to the needs and activities of the pupils, and that geometric facts should be discovered by them. Boys and girls of today are greatly interested in making geometric designs and in determining unknown distances. Boy Scouts and Campfire Girls meet many geometric problems in their experiences. Such problems are real to them and are therefore appreciated and understood. The pupils should be given much drawing and construction work.

The method of study should be that of measuring, drawing, and constructing figures with ruler, compasses, and protractor, and of directed observation. The activities of drawing and measuring call for careful work. They offer excellent practice and training in the use of mathematical instruments. They should be used to lead the pupil to discover experimentally the first geometric principles. Intuition forms the foundation for his first geometric knowledge, and one of the principal aims of the first stage of instruction in geometry is training in space intuition. Acquisition of experiences in the classroom and out of doors, rather than mere memorization of facts, should be the method of absorbing new information, of developing the meaning of essential geometric concepts, and of learning the technical language of geometry. Not too many things are to be attempted, but those matters which every pupil needs to understand are to be thoroughly considered.

In the traditional seventh- and eighth-grade courses some geometry has always been taught under the topic "mensuration." Its major purposes were to acquaint the pupil with the properties of the common surfaces and solids, to teach him to compute areas and volumes by means of rules and formulas, and to show how these rules may be applied to practical problems involving surfaces and content. No systematic training in geometry was provided. Little attempt was made to develop clear meanings of the geometric terms and concepts, which occurred in the applications, to

the extent that they could be used later as bases for further study of geometry. That was considered a task to be performed by the high-school teacher of geometry. On the other hand, the high-school teacher completely ignored all information in geometry which his pupils might have received in the earlier grades, on the ground that it was not sufficiently well taught to be acceptable.

Usually the geometry of the junior high school text goes beyond the work in mensuration found in traditional textbooks on arithmetic. It is nearly all of the informational type, and it is generally derived by the methods of observation, construction, or experimentation. Drawings are used freely to make quantitative facts concrete and to illustrate arithmetical and algebraic concepts. This type of geometry is called "intuitive" geometry. Instruction in intuitive geometry is needed for the following reasons:

1. As has been shown in chapter iv, pupils need to know a number of geometric facts and principles in the study of other school subjects which are ordinarily taught during the junior high school period. Hence, the teaching of such facts and principles during that period is justified. The geometry needed by pupils in other school subjects is well represented in the newer junior high school texts.

2. A certain type of geometry is helpful to the pupil in his everyday life as an aid in understanding his environment. He meets situations requiring power to estimate and to make comparisons of lengths, areas, or volumes. He must be able to appreciate and understand simple designs, maps, and drawings, and he must acquire sufficient skill in the use of the instruments of geometry to be able to make simple diagrams. These abilities are therefore being developed in the newer textbooks.

3. From the standpoint of demonstrative geometry, a preliminary course in intuitive geometry is needed. These needs are easily checked and verified by actual classroom experience. In the new textbooks there is evidence of attempts to prepare pupils for the further study of geometry. As the pupils pass through the seventh, eighth, and ninth grades, they acquire a body of geometric facts in their everyday experiences outside of school even when no instruction in geometry is offered. Moreover, owing to differences in experiences, they will vary in the amount of information which they possess. Inventory tests show that from year to year the body of

known geometric facts increases and that individuals of the same grade differ widely in the amount of geometry which they understand. Hence in the tenth grade the teacher of plane geometry is confronted with a very serious teaching difficulty. He finds it necessary for the sake of a few to present in detail some very elementary work because some pupils have acquired but little and have only partial understanding of that. During this time the other pupils of the class are likely to lose interest in work which seems to them too ridiculously simple. The solution of the problem lies in systematic instruction in intuitive geometry in each grade of the junior high school, and in assigning the task of teaching "logical" geometry to the senior high school. Thus one of the purposes of intuitive geometry should be to prepare the pupil for the logical geometry of the senior high school.

The second stage of instruction in geometry.—It has been shown that Egyptian geometry was concerned with designing and decorating and with the practical geometry employed in surveying and engineering. The time came when Greek mathematicians became interested in the subject. With Thales (640–548 B.C.) there occurred an important change. Previously, geometric facts had been inferred from observation and verified by measurement. Such concepts as point, line, and surface were concrete objects to the Egyptians. Thales and his followers continued to discover new important geometric facts and relations by the method of intuition and measurement and used them to do practical work. However, they soon applied to geometry the process of pure reasoning. They "abstracted" the notions of "geometric" point, line, surface, and circle. They passed from intuitive to logical geometry. Thales is said to have been the first to conceive the idea of logical proof.

Pythagoras (572–501 B.C.), the best known of the followers of Thales, separated geometry from its practical uses and raised it to the rank of a science. By process of reasoning he was able to establish the truth of the theorem relating to the sum of the angles of a triangle, of the relation between the sides of the right triangle, and of some facts about parallel lines and congruent triangles. He also added to the number of geometric constructions. However, the theorems were still studied as separate principles without relation to each other. The logical proofs served as a check to experience.

Hippocrates (*ca.* 460 B.C.) began to arrange the propositions of geometry in scientific fashion. Hypotheses were stated definitely, and every auxiliary theorem was considered in detail.

Finally, Plato (429–348 B.C.) began a systematization of geometric methods and arrangement, thus placing geometry on a logical basis. Emphasis was now given to accurate definitions, clear assumptions, and careful logic in reasoning. Plato had no interest in the practical uses of geometry. Facts were being established and discovered without the use of physical measurement and independently of the degree of accuracy with which the figures were drawn.

The development of geometry from the time of Thales to that of Plato suggests that from the first stage of intuitive geometry the pupil should pass into a second, which is of the nature of a transition from the method of intuition to that of logical rigor and arrangement. However, the learner should not pass abruptly from one into the other. He should gradually come to recognize the advantage and value of a method which is free from inaccuracy and unreliability of measurement. There should come a time when he is impressed by the tediousness and limitations of the intuitive method. Then the logical element should begin to receive attention. He begins to reason about things in an informal way.

Care must be taken that rigor is not carried beyond the point where he is unable to appreciate and understand it. It takes time to prepare him to take up strictly logical demonstrations. For the fact must not be overlooked that the change from intuitive to logical geometry is an exceedingly difficult one. It should be remembered that it took the Greeks three hundred years to transform the practical geometry of the Egyptians into the logical system which was given to the world by Euclid.

The third stage of instruction in geometry.—The third and last epoch begins with Euclid (300 B.C.). He left behind intuitive geometry and undertook to fit together the known facts of geometry into a compact logical system. He insisted on demonstrating every principle. Even the constructions had to be proved before they could be used in a proof. The result was a systematic arrangement of definitions, axioms, postulates, and propositions, and a logical sequence of theorems and problems.

Euclid aimed to proceed logically from a few definitions and axioms and to reach by steps the farthest limits of the study. Demonstrative geometry as taught today is practically the geometry of Euclid. It disregards and fails to provide the important stages of development before Euclid. This accounts to a large extent for the difficulties encountered by the pupils beginning the study. They are not sufficiently prepared to enter the third stage and to take up the study of logical geometry unless they have previously developed an appreciation and understanding of the fundamental concepts of geometry and of the meaning of logical proof.

If intuitive geometry precedes logical geometry, it will be possible to assume as true all the principles established in the former work. It will be essential that the pupil know what the assumptions are, and that he distinguish clearly between an informal discussion and a rigorous proof—a distinction which is not always made in demonstrative geometry.

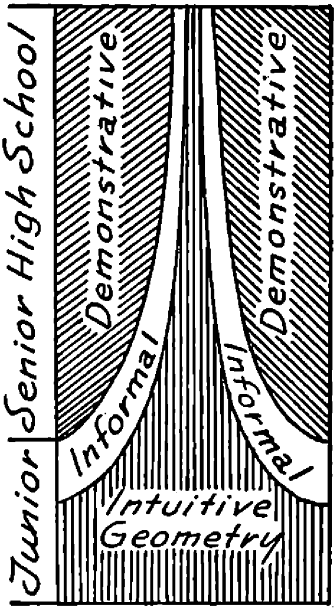


FIG. 18.—Graphic representation of the transition from intuitive to logical geometry.

The distribution of the three stages of geometry taught during the secondary-school period is illustrated in Figure 18. It shows that in the beginning geometry might be entirely intuitive. Gradually, informal reasoning may be stressed. In turn it will give way to demonstrative geometry. However, intuition and informal reasoning are never to be entirely dropped.

For the first stage such content should be selected as the pupil actually needs in his everyday life, particularly in the other school subjects. It should be useful to him and practical, falling within his actual experiences and aiding him to understand his environment. It should enable him to meet geometric situations in school and outside of school. Such activities and experiences should be provided as are real to him. Junior high school geometry is the ge-

ometry of the classroom where pupils observe directly and discuss points, lines, angles, and other geometric figures. It is the geometry of other school subjects, such as shopwork, sewing, geography, and science, in which pupils are called upon to make diagrams, designs, maps, floor plans, and graphs. It is the geometry of out-of-door life which enables boys and girls to solve problems in inaccessible distances, such as finding heights of buildings, or distances across streams. The textbook is merely suggestive. The pupils themselves should bring to the classroom local materials and problems, which either replace textbook material entirely or at least supplement it.

Junior high school geometry must comprise the fundamentals of the geometry of the vocational mathematics as well as those of future mathematical courses.

It must help the pupil acquire clear understanding of the elementary concepts of plane geometry and notions of space relations, and it must put him in possession of fundamental geometric principles. It must develop in the pupil understanding, appreciation, fundamental skills, right habits of work, and effective methods of thinking.

The method should be that of induction, observation, and measurement. The pupil should develop skill in the use of the drawing instruments.

The instructional geometric materials must be arranged in psychological and pedagogical sequence. There should be a gradual transition from the method of intuition to that of logical demonstration, bridging the gap between the present geometry of the elementary school and the later geometry of the high school.

Time and effort must be economized by adapting the course to the general nature, capacity, and mental growth of the learner, by avoiding needless repetitions, and by keeping in constant use new ideas and facts once acquired.

The problem of organizing the content of intuitive geometry.—If the principle is accepted that intuitive geometry should be a study required of all pupils, it still remains to decide at what time of school life it should be assigned a place in the curriculum, and how to organize the content for teaching purposes.

Examination of textbooks on junior high school mathematics reveals a tendency to include a steadily increasing quantity of geo-

metric content in the early grades. However, there is a decided lack of agreement as to the amount of time to be given to the various topics. For example, an examination of different textbooks has shown that the number of pages devoted to the study of areas and volumes varies from 1 to 50, or even more. Furthermore, it was found that a particular topic is being treated in any one of the junior high school grades, and that there is no agreement as to the grade in which it should be presented.

For organizing the instructional materials of a course in intuitive geometry the following procedure is recommended:

1. Tentative units of instruction should be selected.
2. The instructional materials should then be distributed among the various units.
3. In each unit the geometric facts and principles should be arranged in order of difficulty, the order being determined by a testing program devised for that purpose.
4. The content should then be taught to classes and further tests given at the end of each unit. On the basis of the results of the tests the content of units should be repeatedly revised and retaught until a satisfactory arrangement is reached.

The following is a list of the central themes of units of intuitive geometry that have been tried out in the laboratory schools of the University of Chicago:

a) The line segment
b) The angle
c) The circle
d) The graph
e) Indirect measurement
f) Areas of plane figures
g) Areas and volumes of the common solids
h) Geometric constructions

It is not to be inferred from the list that each concept must be developed entirely by itself to the exclusion of all the others, but rather that the content of each unit should center around the particular concept which thereby is given especial emphasis.

In organizing the content of geometry the teacher may have to make important changes in the traditional course. In doing this he should keep in mind the aims and purposes of the subject and the adverse criticisms which have been made. The following recommendations will go far toward eliminating criticisms:

1. The organization should make provision for the study of geometry as part of the first year's work of the secondary school and

continue to provide systematic instruction in geometry in the various mathematical courses which follow. The slow progress will improve the pupil's chances to acquire a thorough understanding of the fundamental geometric concepts.

2. Emphasis should be on the practical phases of the subject and on geometric information which pupils need in other subjects and in everyday activities.

3. During the first years of instruction geometry of three dimensions should not be separated from two-dimensional geometry. Pupils should manipulate solids, and should observe plane figures on models, in the classroom and out of doors.

4. Intuitive geometry should gradually change to the formal logical geometry and thus bridge the gap between the two types.

5. The arrangement of subject matter should be psychological, i.e., materials should be arranged according to difficulty and the way they are most easily and economically learned. Difficulty and ease of learning are to be determined by a careful testing procedure as outlined in chapter ii.

For a period of years experiments were conducted in the laboratory schools of the University of Chicago with a variety of arrangements of geometric subject matter. The organization which was the outcome of the investigation and which most nearly satisfied the foregoing requirements will be described in the next pages.

The three levels of intuitive geometry.—Intuitive geometry was distributed according to three levels: geometry of length, geometry of areas, and geometry of volumes. The second and third levels are planned to review and to keep in use the work of the preceding. The content was taught to pupils of Grades VII, VIII, and IX. Emphasis in instruction was on observation and experimentation. However, toward the end of the three-year period a transition was made to the more formal type of logical geometry. Thus, the need for the formal proof of geometric facts was aroused long before the pupil reached the tenth grade in which demonstrative geometry is traditionally taught.

Assimilative materials for the first level.[5]—In Table XXVI a detailed account is presented of the first level of intuitive geometry.

[5] For additional details the reader is referred to *Seventh-Year Mathematics* and *Eighth-Year Mathematics*.

TABLE XXVI

CONTENT OF THE FIRST LEVEL OF INTUITIVE GEOMETRY; PUPIL ACTIVITIES, APPLICATIONS, AND ATTAINMENTS

Activities	Applications	Attainments
	The Line Segment	
1. Observing lines....	In the classroom, on the athletic field, on models, in geometric diagrams.	Understanding of the meanings of line and line segment. Ability to recognize lines in a variety of situations.
2. Drawing lines.....	In making simple diagrams, scale drawings, and graphs.	Ability to use the drawing instruments and to make simple diagrams.
3. Measuring line segments by using the ruler, compasses, and squared paper.	In making objects in the shop, floor plans, diagrams, in scale drawings, and graphs; in boundaries of land, in maps; in finding perimeters.	Ability to use drawing instruments, to approximate length, and to use the metric and English system of measures. Knowledge of the names of polygons, of the meaning of isosceles and equilateral triangles, of the perimeter formula.
4. Denoting line segments by letters...	In making and reading diagrams. In finding perimeters.	Ability to use algebraic notation in discussions relating to lines.
5. Comparing line segments...........	In interpreting graphs. In finding ratios of line segments.	Ability to find the ratio of two line segments.
	The Angle	
1. Observing angles..	In the classroom, on the playground, in geometric figures, and on models.	Understanding of the meaning of angles. Ability to recognize angles.
2. Denoting angles by symbols..........	In reading diagrams. In designs.	Ability to use various ways of denoting angles in reading and discussions.
3. Classifying angles.	Recognizing angles of different sizes in the classroom, in diagrams.	Meaning of the concepts acute, right, obtuse, and straight angle.

TABLE XXVI—*Continued*

Activities	Applications	Attainments
	The Angle—*Continued*	
4. Measuring angles..	In diagrams, in the classroom, on buildings, in interpreting blueprints, and in house plans.	Understanding of "size" of angles. Ability to use the protractor. Ability to estimate the sizes of angles. Knowledge of units of angular measure. Meaning of right triangle. Meaning of approximate size of an angle.
5. Drawing angles of given sizes.......	In making designs, in making diagrams, in drawing triangles from given parts, in making floor plans, in pattern-making, in making survey diagrams, in making shop drawings.	Ability to use the protractor. Familiarity with the units of angular measure.
6. Discovering relationships between angles...........	In geometric diagrams, in designs, and in surveying problems.	Meaning of adjacent angles, complementary angles, supplementary angles, perpendicular lines, parallel lines. Knowledge of relations between opposite angles, complementary angles, supplementary angles, the angles of a triangle, the acute angles of a right triangle, the base angles of an isosceles triangle, anglés formed by parallel lines, and transversals, and angles formed by perpendicular lines. Ability to formulate theorems from observation and measurement.
	The Circle	
1. Observing circles..	In architecture, designs, machinery, and measuring devices.	Appreciation of the usefulness of circles. Ability to recognize circles. Meaning of radius, diameter, and arc.

TABLE XXVI—*Continued*

Activities	Applications	Attainments
	The Circle—*Continued*	
2. Drawing circles with compasses...	In making designs involving equilateral triangles and squares.	Skill in the use of compasses. Ability to make good drawings involving circles.
3. Solving problems related to circles..	In computing gas and electric-light bills. In latitude and longitude problems.	Ability to make good freehand drawings of circles. Appreciation of the degree of accuracy in solving problems.
4. Finding the circumference of a circle............	In problems with circular objects, as lamp shades and flower beds.	Appreciation of the relationship between radius and circumference. Knowledge of the circumference formula. Ability to solve problems. Understanding of approximate measurement.
	The Graph	
1. Interpreting picture graphs.......	In population reports, production of farm products, and geography.	Meaning of graphical representation. Ability to interpret picture graphs.
2. Representing numbers by line segments...........	In scale drawings of the classroom, school yard, and school building.	Ability to round off numbers, to read and understand floor plans.
3. Making and interpreting bar graphs	In statistics on population, farm crops, heights of buildings, profits, sales, and food prices.	Ability to round off large numbers, to understand and to make bar graphs.
4. Making and interpreting line graphs	In picturing changes in population, temperature, rainfall, growth of pupils, school attendance, and pupils' marks.	Ability to understand line graphs. Appreciation of relationships.
5. Making and interpreting circular graphs...........	In statistics on populations, budgets, distribution of incomes, taxes.	Ability to interpret circular graphs.

It is concerned with measurement of lines and angles. Form study is carried on in a systematic way and gives the pupil a basis for later scientific study. When he understands clearly geometric concepts and geometric drawings, the first of the major difficulties in the study of geometry is removed. The activities listed in the table are not to be separated or carried on in the order shown by the numerals. There will necessarily be much overlapping of activities.

The pupil is taught to make simple analyses of figures on the basis of direct observation. They become the foundation for the more elaborate discussions and analyses of plane geometry. Thus another difficulty encountered by the pupil in the study of logical geometry is removed.

THE FUNDAMENTAL CONSTRUCTIONS

The fundamental constructions are:

1. Bisecting an angle
2. Bisecting a line segment
3. Constructing a perpendicular to a line
   a) At a point on the line
   b) From a point not on the line
4. Constructing an angle equal to a given angle

The fundamental constructions are to be made with ruler and compasses. Each is applied to a variety of problems in designs, to circular graphs, and to more complicated construction problems. It is aimed to develop the concepts of symmetry in geometric figures, inscribed and circumscribed regular polygons, distance from a point to a line, and tangent to a circle.

The fundamental constructions are used in the following problems:

1. Constructing an equilateral triangle
2. Constructing an angle of 60°
3. Dividing a circle into six equal parts
4. Constructing a regular hexagon
5. Constructing the altitudes, medians, bisectors of angles, and perpendicular bisectors of the sides of a triangle
6. Constructing parallel lines
7. Constructing circular graphs

The assimilative materials for the second level.—Table XXVI has shown the organization of instructional materials of the first level of intuitive geometry. It is concerned largely with measurement of lines and angles. The second level deals with the measurement of

TABLE XXVII

CONTENT OF THE SECOND LEVEL OF INTUITIVE GEOMETRY

Activities	Applications	Attainments
	Areas and Properties of Plane Figures	
1. Measuring sides and angles of a rectangle. Drawing rectangles	In planning flower beds for gardening and computing lighting surface in the classroom.	Acquaintance with the important properties of the rectangle. Ability to draw rectangles with ruler, compasses, and protractor.
2. Measuring the surface bounded by a rectangle: *a*) By counting unit squares *b*) By formula	In problems of surveying of the playground, of purchasing carpets and rugs. In problems occurring in the trades.	Understanding of the meaning of area. Ability to measure surfaces of rectangles by means of squared paper, or by use of the formula. Ability to use the area formula in problems.
3. Measuring the sides, angles, and surface of the square. Drawing squares	In picturing percentage. In problems involving square measure.	Ability to find the area of a square by means of squared paper. Ability to use the formula for the area of the square in problems.
4. Studying the relations between the sides of the right triangle	In problems of finding unknown distances.	Understanding of the relation between the squares on the sides of a right triangle, and the relation between the sides.
5. Measuring sides, angles, and area of the parallelogram	In making designs. In computing land measure.	Ability to draw parallelograms. Understanding of the formula for finding the area of the parallelogram. Acquaintance with the important properties of parallelograms.
6. Measuring surfaces bounded: *a*) By the trapezoid *b*) By the triangle *c*) By the circle	The uses are similar to those in 1–5 above. An abundance of practical problems is worked out.	Understanding of the meaning of area, and of relationships between parts of the figures studied. Appreciation of the uses of the figures. Familiarity with the properties of figures. Ability to measure and draw. Ability to use the formulas in problems. Skill in the use of drawing instruments.

the surfaces of plane figures. The content is presented topically in Table XXVII.

The third level, or three-dimensional geometry.—The study of solid geometry introduces a difficulty which does not appear with two-dimensional figures. Many pupils are unable to picture three-dimensional figures in space by examining the flat figures in the textbook. Until they can do this they cannot proceed intelligently. Some teachers and textbooks assist the learner by using models and photographs. Others object to them on the ground that they are crutches, that they interfere with reasoning, and that they hinder the development of spatial conception.

Intuitive geometry aims to solve the problem by developing space imagery before the pupil reaches the stage of logical geometry. It avoids the mistake of limiting the training of the pupil to two-dimensional geometry to the exclusion of three-dimensional figures, as is the case in the traditional courses in demonstrative geometry. Even in the first and second levels the pupil's attention is constantly called to two-dimensional figures in three-dimensional space. From Table XXVII it is seen how the pupil is directed to three-dimensional space for illustrations of the diagrams he is studying. To this should be added the study of three-dimensional geometry, the content of which is outlined in Table XXVIII. The pupil makes his own models. He studies them and examines them to determine their characteristic properties. The last step is to make drawings to represent three-dimensional figures, and to study them instead of the models.

The transition from the first to the second stage of geometry.—Experimentation has shown that best results cannot be obtained by an abrupt change from the stage of intuition to the stage of informal reasoning and from there to strictly logical reasoning. As the pupil develops reasoning powers he must be given material on which to try his skill. As soon as he finds reasoning easy and advantageous he is more than anxious to drop measurement as a method of proof and to reason about things.

Thus, in the second stage logic and space are gradually joined. The geometry of intuition is supplemented, and sometimes replaced, by informal reasoning which gives training in the use of simple analyses, points to the need and meaning of logical proof,

TABLE XXVIII

CONSENT OF THE THIRD LEVEL OF INTUITIVE GEOMETRY

Activities	Applications	Attainments
	Properties, Areas, and Volumes of the Common Solids	
1. Making models of rectangular blocks, cubes, cylinders, prisms, cones, and pyramids	In problems relating to objects found in the pupil's daily activities and experiences in and out of school, such as boxes, silos, church steeples.	Understanding of the forms of the common solids. Ability to recognize them in other objects.
2. Studying the properties of solids, including the sphere	In buildings, ornaments, and numerous objects, such as boxes and rooms. In spherical objects, in geography, in science.	Ability to see relationships between lines and planes in space. Understanding of relationships between the parts of a solid. Understanding of new terms and concepts, such as vertex, edge, face, and altitude. Ability to visualize a sphere from a drawing.
3. Making drawings of the solids	In various school subjects. In illustrations of problems.	Appreciation of space relations. Ability to understand drawings of solids. Skill in making free-hand drawings of solids.
4. Developing formulas for finding areas and volumes of the solids, and using them in problems	In problems involving the common solids.	Ability to use the formulas in geometric-problem situations. Familiarity with units of volume, including the metric units.
5. Using the units of measure of volume	In measuring the content of objects, such as pyramids, cylinders, and cones.	Ability to solve problems in which the units of volume are used.

and paves the way for the study of situations that are more difficult than those of the earlier stages. The first two stages teach the geometry which people in general should know. They should therefore be required of all pupils.

To increase the opportunity for making the transition to the third stage and to satisfy the pupil's geometric needs in and out of school, a body of geometric materials is outlined in Table XXIX. It gives a new view of the geometry studied previously and extends further what has been taught. As in the previous outlines, the numerals in the table do not indicate the sequence in which the activities are to occur and the reader is not to imply that the activities are strictly separated from each other.

Table XXIX shows numerous opportunities for using and reviewing facts about triangles and circles; for practice in measuring and drawing line segments and angles with ruler, compasses, and protractor; and for establishing clear meanings of the important geometric concepts. Furthermore, it is seen that the study of geometry is being considerably extended by the inclusion of the ideas of congruence and similarity. The objections to the traditional proofs of the congruence theorems are thereby eliminated. These theorems may later be listed among the assumptions in logical geometry.

Articulation of intuitive and logical geometry.—Logical geometry, as traditionally taught, begins with the last and most advanced stage of the development of geometry. Not enough time is allowed to develop clear meanings of new concepts. Usually they are derived by definitions rather than from experiences. The type of reasoning employed in logical proof is new to the pupil and therefore not easily understood. The first principles to be proved are generally known to many pupils, and they do not understand why proofs should be required. Confusion arises in the minds of the learners, and they are likely to develop an unfriendly attitude toward the subject.

Intuitive geometry offers ample opportunity for providing the introductory stages to the study. The method is that of observation, comparison, and measurement. It allows sufficient time, three years if necessary, for broad experiences and activities from which the meanings of the important geometric concepts may be derived.

To the seventh-grade pupil the fundamental principles of geometry are new, and it gives him pleasure to discover them. He develops

TABLE XXIX

Activities	Applications	Attainments
	Indirect Measurement	
1. Drawing triangles of the same size and shape having given: *a*) Two sides and the included angle *b*) Two angles and one side	Finding distances and angles indirectly by congruent triangles. Determining lengths of lines passing through a building, through a hill, across a swamp. Finding heights of trees, smokestacks, and flagpoles.	Acquaintance with the surveying instruments. Understanding of the concept of congruence. Knowledge of two theorems on congruent triangles: s.a.s. and a.s.a. Skill in making constructions with ruler and compasses. Ability to measure lines and angles.
2. Making scale drawings of triangles, rectangles, and other figures......	Finding distances and angles indirectly by scale drawings. Making scale drawings of designs.	Understanding of scale drawing, designs, and blueprints. Meaning of the terms "angle of elevation" and "angle of depression." Ability to use squared paper for making scale drawings.
3. Drawing triangles similar to given triangles: *a*) If the angles of one are equal to the angles of the other *b*) If the ratios of the corresponding sides are equal	Finding distances and angles indirectly by means of similar triangles. Solving Scouts' problems in surveying.	Understanding of similarity of triangles. Knowledge of two theorems on similar triangles.
4. Finding unknown distances by use of the table of tangents......	Problems in surveying, such as finding the height of a flagpole and the angle of elevation of a road.	Ability to understand and use the table of tangents in finding unknown distances.

skill in the use of the mathematical instruments. One of the major functions of intuitive geometry is to secure continuity in instruction

by providing a natural transition to demonstrative geometry and to create interest in logical geometry.

If a systematic course in intuitive geometry is offered in all seventh-, eighth-, and ninth-grade classes, the teaching of demonstrative geometry will be tremendously simplified. It will only be necessary to take an inventory of what the pupils actually know. Instruction in geometry may then begin at the point where the junior high schools have left off. Unfortunately, the situation is still far from ideal, because junior high school teachers do not agree as to how much and how effectively geometry should be taught in the early grades. Hence, the teachers of demonstrative geometry have to teach classes in which the pupils vary widely in the type of instruction they have received and in the amount of subject matter actually understood and retained. Many teachers are thereby led to disregard entirely the work of the lower school and to start from the very beginning. The results, as may be expected, are not satisfactory. Pupils cannot be enthusiastic about repeating subject matter previously mastered. The teacher who attempts it will find lack of interest and dissatisfaction. On the other hand, pupils do not object to proving in a new way theorems previously established by intuition. However, it must be made clear to them that the purpose is not to convince them of the truth of facts that are known but to show new ways of deriving them. This is frequently done in logical geometry and experience shows that pupils enjoy it. When an exercise is assigned to a class more than one solution or proof should be discovered and presented. Indeed, it would be disappointing if all the pupils of a class would always choose the same method of proof. Interest in some of the famous theorems, e.g., the theorem of Pythagoras, is increased by proving them in many ways and by including them in several of the units of the course.

Until the work in geometry in the junior and senior high school is carefully articulated there will be waste, and progress will be retarded.

Purposes of demonstrative geometry.—Although the content of demonstrative geometry possesses much informational value, the major purposes of the course are the cultivation of reasoning powers and appreciation of the logical demonstration and system. The pupil should be trained to make discoveries by elaborate logical

processes of analysis, comparison, and drawing of inferences. However, experimentation has shown that even at this stage it would be a mistake to neglect space observation entirely and to emphasize logic alone. When this is done the pupil receives too little training in space observation, and frequently he does not even associate the figures in his textbook with others that he observes about him in his everyday experiences.

Furthermore, it will be seen from the outlines given in the following pages that the amount of time given to the third stage of geometry may be reduced considerably below that required for the traditional course in plane geometry without any loss to the learner. For all facts previously established may now be assumed as true. The time thus gained may be used to enrich the course by introducing materials from solid geometry and trigonometry.

The technique used in arranging the geometric content for the third stage.[6]—As for the first two stages, tentative units of instruction were selected. They were arranged in the order in which it was intended to present them, and were then tried out in high-school classes. On the basis of the experience gained, the list was revised, retaught, and again revised. The following is a list of central themes of the units derived by this procedure. It is not to be inferred that the order in which they are listed is necessarily the order in which they are to be taught.

I. Angle relations formed by parallel and perpendicular lines
II. Quadrilaterals
III. Relations between line segments intercepted by parallel lines and transversals
IV. Loci and concurrent lines
V. Similar figures and relations between the parts of triangles
VI. The circle
VII. Inequalities
VIII. Regular polygons and the circle
IX. Areas

The various geometric facts and principles that are usually taught in plane geometry were distributed among the foregoing nine divisions. In each division the content was then classified as follows:

[6] For complete description of the various units see *Senior Mathematics*, Book II.

1. New terms and concepts to be established by the methods of intuition, measuring, and drawing. Concepts are to be illustrated in the classroom in three-dimensional space.

2. Facts too evident to the high-school pupil to require proof. They are to be established by intuition and experimentation.

3. Simple facts and theorems that are easily established by informal reasoning and discussion.

4. Theorems that require logical proofs and are especially adapted to furnish training in reasoning. They are to be arranged in logical sequence and to be demonstrated by the methods of proof usually designated as analysis, indirect, congruence, and similarity.

5. Theorems that may be treated as exercises and may be established by either informal reasoning or logical proof.

6. Social and practical applications necessary to help the pupil assimilate the principles of the unit, and suitable to give training in methods of attacking and solving original exercises.

7. Materials that lend themselves particularly to algebraic manipulation, and give training in the use of algebraic notation, in manipulation, and in solving equations.

8. Materials that give training in arithmetical computation.

9. Exercises and theorems that offer opportunity for training in functional thinking.

10. Constructions to be made with ruler and compasses.

It will be noted that the procedure reduces considerably the number of theorems to be proved by the logical process, since all of the facts established in the lower courses and some new facts that may be established by the methods of intuition and informal reasoning are assumed without proof.

Outlines of the principal facts to be taught in the various divisions follow. For teaching purposes the content should be classified and presented according to the foregoing ten types of materials, but it will have to be arranged in logical sequence.

I. ANGLE RELATIONS FORMED BY PARALLEL AND PERPENDICULAR LINES

A. Development of the meaning of parallel and perpendicular lines, transversal, diagonal, converse of theorem, exterior angle of polygon, alternate interior angles, and alternate exterior angles

B. Theorems to be discussed informally
   *1. Through a given point not on a given straight line there is one and only one line parallel to the given line.[7]
   ⊛2. The sum of the angles of a triangle is 180°.
   3. The sum of the interior angles of a polygon of n sides is $(n-2)$ straight angles.

C. Theorems to be proved logically
   1. If two lines are parallel to a third line they are parallel to each other.
   ⊛2. From a point outside of a given line only one line can be drawn perpendicular to the given line.
   3. Two lines perpendicular to the same line are parallel.
   ⊛4. If two lines are cut by a transversal making the alternate interior angles equal the lines are parallel.
   5. Two lines are parallel if the corresponding angles formed with a transversal are equal.
   6. Two lines are parallel if the "interior angles on the same side" formed with a transversal are supplementary.
   ⊛7, 8, and 9. The converse theorems of 4, 5, and 6.
   *10. If one of two parallel lines is perpendicular to a third line, the other is also.

D. Exercises and applications to angles, triangles, quadrilaterals, and other polygons, to be demonstrated by algebraic methods
   1. If two angles have their sides parallel, they are either equal or supplementary.
   2. If two lines are cut by a transversal, they are parallel if the corresponding angles are equal, or if the interior angles on the same side are supplementary.
   3. The converse theorem of 2.
   4. The sum of the exterior angles of a polygon is 360°.

E. Direct and indirect methods of proof are to be taught

F. Constructions
   1. Through a given point to draw a line parallel to a given line.
   2. To draw a line perpendicular to a given line and passing through a given point on the given line.
   3. To draw a line perpendicular to a given line and passing through a given point not on the given line.

[7] A theorem marked with an asterisk (*) is found in the list of the College Entrance Board, one marked with a circle (⊛) is found in the list of the National Committee on Mathematical Requirements.

II. THE QUADRILATERALS

A. Development by intuition and measurement of the meanings of quadrilateral, parallelogram, rectangle, square, trapezoid, isosceles trapezoid, and rhombus

B. Theorems to be assumed, if known, or to be proved in full, or to be treated as exercises

a) Important properties of a parallelogram

⊛1. The diagonal divides the parallelogram into two congruent triangles. [*Prove.*]

2. The opposite sides of a parallelogram are equal. [*Exercise.*]

3. The opposite angles of a parallelogram are equal. [*Exercise.*]

4. The consecutive angles of a parallelogram are supplementary. [*Exercise.*]

5. The diagonals of a parallelogram bisect each other. [*Prove.*]

b) Important properties of quadrilaterals

1. The sum of the interior angles of a quadrilateral is 360°. [*Exercise.*]

2. The sum of the exterior angles of a quadrilateral is 360°. [*Exercise.*]

c) Conditions which make a quadrilateral a parallelogram

⊛1. If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram. [*Prove.*]

⊛2. If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram. [*Prove.*]

3. If the diagonals of a parallelogram bisect each other, the quadrilateral is a parallelogram. [*Prove.*]

4. If the opposite angles of a quadrilateral are equal the quadrilateral is a parallelogram. [*Prove.*]

C. Constructing a parallelogram if two adjacent sides and the included angle are given

D. Applications of various theorems to the square, rectangle, trapezoid, and rhombus; practical applications; the parallelogram of forces

E. Exercises employing the algebraic processes of elimination which are used in some of the proofs, and the solutions of linear equations

III. RELATIONS BETWEEN LINE SEGMENTS FORMED BY PARALLEL LINES AND TRANSVERSALS

A. Development of the meaning of the terms: unit segment, measure of a segment, ratio of two segments, proportional line segments, median of a trapezoid, alternation, and inversion

B. Theorems

⊛1. If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal. [*Prove.*]

2. If a line bisects one side of a triangle and is parallel to a second, it bisects the third side and is equal to one-half of the second. [*Exercise.*]

⊛3. If three or more parallel lines are cut by two transversals, the ratio of the segments on one transversal is equal to the ratio of the corresponding segments of the other. [*Prove.*]

⊛4. The converse theorem of 3. [*Prove.*]

⊛5. If a line is parallel to one side of a triangle, it divides the other two sides proportionally. [*Exercise.*]

°6. If a line divides two sides of a triangle proportionally, it is parallel to the third side. [*Prove.*]

7. A line joining the midpoints of two sides of a triangle is parallel to the third side. [*Exercise.*]

⊛8. The bisector of the interior angle of a triangle divides the side opposite the angle into segments proportional to the other sides. [*Prove.*]

⊛9. The bisector of the exterior angle of a triangle divides the opposite side externally into segments proportional to the other two sides. [*Exercise.*]

C. Constructions

1. To divide a segment into equal parts.
2. To divide a segment in a given ratio.
3. Construct a segment proportional to two or three given segments.

D. Applications of the theorems to the triangle, trapezoid, and other quadrilaterals

E. Algebraic processes involving proportions

1. Solving equations of the form of proportions.
2. Forming proportions from two equal products.

IV. LOCI AND CONCURRENT LINES

A. New terms and concepts: locus, concurrent lines, inscribed circle, circumscribed circle, tangent to a circle, median of triangle, altitude of triangle, and perpendicular bisector of the side of a triangle

B. Constructions

1. To construct a circle circumscribed about a triangle.
2. To construct a circle inscribed in a triangle.
3. To construct the medians of a triangle.

4. To construct the altitudes of a triangle.
5. To construct the perpendicular bisectors of the side of a triangle.

C. Theorems on loci

⊛1. The perpendicular bisector of a segment is the locus of points equidistant from the end points. [*Prove.*]

⊛2. The bisector of an angle is the locus of points equidistant from the sides. [*Prove.*]

D. Theorems referring to concurrent lines

1. The bisectors of the interior angles of a triangle are concurrent. [*Prove.*]
2. The perpendicular bisectors of the sides of a triangle are concurrent. [*Prove.*]
3. The altitudes of a triangle are concurrent. [*Prove.*]
4. The medians of a triangle are concurrent. [*Prove.*]

E. Applications to loci in two dimensions and in three dimensions

V. SIMILAR FIGURES

A. New terms and concepts: similarity of figures, projection, and mean proportional

B. Theorems relating to similar triangles

⊛1. Two triangles are similar if two angles of one are equal to two angles of the other. [*Prove.*]

⊛2. Two triangles are similar if an angle of one is equal to an angle of the other, and if the sides including the equal angles are proportional. [*Prove.*]

⊛3. Two triangles are similar if the sides are proportional. [*Prove.*]

4. The perimeters of similar triangles are proportional to any two corresponding sides. [Exercise.]

5. A line parallel to a side of a given triangle forms with the other two sides a triangle similar to the given triangle. [*Prove.*]

C. Theorems referring to similar polygons

°1. The perimeters of similar polygons are proportional to any two corresponding sides. [*Exercise.*]

⊛2. Similar polygons can be divided into similar triangles by drawing diagonals from two corresponding vertices to the other vertices. [*Prove.*]

D. Applications of the theorems on similar triangles

a) Theorems relating to the right triangle

°1. The perpendicular to the hypotenuse from the vertex of the right angle of a right triangle divides the triangle into parts similar to each other and similar to the given triangle. [*Prove.*]

2. The perpendicular from the vertex of the right angle to the hypotenuse is the mean proportional between the segments of the hypotenuse. [*Prove.*]
3. Either side of the right angle of a right triangle is a mean proportional between its projection upon the hypotenuse and the entire hypotenuse. [*Prove.*]

°4. The square on the hypotenuse is equal to the sum of the squares of the sides of the right angle. [*Prove.*]

b) Theorems relating to the oblique triangle

1. The square on the side opposite an acute angle is equal to the sum of the squares on the other two sides diminished by twice the product of one of them by the projection of the other on it. [*Prove.*]
2. The square of the side opposite an obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of them by the projection of the other on it. [*Prove.*]

c) Applications to problems in surveying

E. Constructions

1. Construct a mean proportional between two line segments.
2. Construct the square root of a number.
3. Construct a polygon similar to a given polygon.

F. Algebraic skills

1. Solving problems leading to proportions.
2. Simplifying radicals of the form $\sqrt{ab}$.
3. Solving quadratic equations.

VI. THE CIRCLE

The unit is divided into three parts: (1) relationships between chords, secants, tangents, and arcs; (2) relationships between angles and arcs; and (3) relationships between segments of chords and tangents.

A. Relationships between chords, tangents, secants, and arcs

a) Meaning of the terms: circle, radius, diameter, center, arc, secant, tangent, chord, semicircle, concentric circles, and tangent circles

b) Theorems established by intuition

1. Circles having equal radii on equal, and equal circles have equal radii.
2. A diameter divides a circle into equal parts.

°3. In the same or equal circles equal central angles intercept equal arcs, and equal arcs are intercepted by equal central angles.

c) Theorems to be established by logical proof

°1. In the same or equal circles equal arcs are subtended by equal chords, and the converse theorem.

°2. A line drawn through the center of a circle perpendicular to a chord bisects the chord and the arcs subtended by the chord.

Other theorems dealing with the same facts may be proved or discussed as exercises.

⊛3. In the same or equal circles equal chords are equally distant from the center, and chords equally distant from the center are equal.

4. The arcs included between parallel secants are equal; if two secants cut off equal arcs and do not intersect within the circle, they are parallel.

5. The line joining the centers of two circles bisects the common chord perpendicularly.

°6. The radius drawn to the point of tangency is perpendicular to the tangent, and a line perpendicular to a radius at the outer extremity is tangent to the circle.

7. If two circles are tangent to each other, the centers and the point of tangency lie in a straight line.

*8. Through three points not lying in a straight line one circle and only one can be drawn.

B. Relationships between angles and arcs

a) Meanings of the terms: intercepted arc, inscribed angle, inscribed and circumscribed polygon, and segment of a circle

b) Theorems to be established by intuition

1. A central angle is measured by the intercepted arc.

°2. In the same or equal circles the ratio of two central angles is equal to the ratio of the intercepted arcs.

c) Theorems to be proved logically

⊛1. An inscribed angle is measured by one-half the intercepted arc.

2. If two chords intersect, either angle formed is measured by one-half the sum of the intercepted arcs.

3. An angle formed by a tangent and a chord passing through the point of contact is measured by one-half of the intercepted arc.

4. If two secants meet outside of a circle, the angle formed is measured by one-half the difference of the intercepted arcs.

5. The angle formed by a tangent and a secant meeting outside of a circle is measured by one-half of the difference of the intercepted arcs.

6. The angle formed by two tangents to a circle is measured by one-half the difference of the intercepted arcs.

C. Relationships between segments of chords and tangents

⊛1. If two chords of a circle intersect, the product of the segments of one is equal to the product of the segments of the other.

2. If from a point outside of a circle a tangent and secant are drawn, the tangent is a mean proportional between the entire secant and its external segment.

3. If from a point outside of a circle two secants are drawn to the concave arc, the product of one secant and its external segment is equal to the product of the other secant and its external segment.

D. Constructions

1. Bisect a given circle arc.
2. Given an arc, find the center and draw the complete circle.
3. Construct a circle passing through three points not in the same line.
4. Upon a line segment as a chord construct a circle arc whose inscribed angle is equal to a given angle.

°5. From a point outside of a circle construct a tangent to the circle.

6. Draw the external and internal tangents common to two circles exterior to each other.
7. Divide a segment into mean and extreme ratio.

VII. INEQUALITIES

The study of this material is designated as optional by some writers. Others suggest that the content be distributed over the course. Thus, following the axiom that the sums of equals added to equals are equal, the question may be raised as to the sums when equals are added to unequals. After proving that two sides of a triangle are equal if the angles opposite them are equal, the pupil may be asked to find out how the two sides compare if the angles opposite them are unequal. The theorems on inequalities will answer questions similar to the foregoing.

A. Preliminary assumptions of inequality established by arithmetical examples and by intuition

1. A line segment, or an angle, is greater than a part of itself.
2. The sums obtained by adding unequals to equals are unequal in the same order as the unequal addends.
3. The sums obtained by adding unequals to unequals in the same order are unequal in the same order.
4. If three magnitudes are so related that the first is greater than the second and the second greater than the third, the first is greater than the third.
5. If equals are subtracted from unequals, the remainders are unequal in the same order as the unequal minuends.

6. The differences obtained by subtracting unequals from equals are unequal in the order opposite to that of the subtrahends.
7. The products obtained by multiplying unequals by positive equals are unequal in the same order as the multiplicands.
8. The products obtained by multiplying unequals by negative equals are unequal in the order opposite to that of the multiplicands.
9. The quotients obtained by dividing unequals by positive equals are unequal in the same order as the dividends.
10. The quotients obtained by dividing unequals by negative equals are unequal in the order opposite to that of the dividends.
11. The shortest distance between two points is the straight line segment joining the points.

B. Facts established by informal reasoning
1. The sum of two sides of a triangle is greater than the third side, and the arithmetical difference is less than the third side.
2. The shortest distance from a point to a line is the perpendicular from the point to the line.
3. If two sides of a triangle are unequal, the angles opposite them are unequal, the greater angle lying opposite the greater side.
4. If two angles of a triangle are unequal, the sides opposite them are unequal, the greater side lying opposite the greater angle.
5. Any point not on the perpendicular bisector of a line segment is unequally distant from the end-points.
6. A point not on the bisector of an angle is unequally distant from the sides of the angle.

C. Theorems to be proved
1. The diameter of a circle is longer than any other chord of the circle.
2. An exterior angle of a triangle is greater than either of the remote interior angles.
3. If two oblique line segments drawn to a line from a point on the perpendicular to the line have unequal projections, the oblique line segments are unequal.
4. Two unequal oblique line segments drawn to a line from a point on a perpendicular to the line have unequal projections.
5. If from a point inside a triangle line segments are drawn to the end-points of one side, the sum of the line segments is less than the sum of the other two sides.
6. In the same circle or in equal circles, unequal chords are unequally distant from the center of the circle, the shorter chord lying at the greater distance; and, conversely, chords unequally distant from

the center are unequal, the chord at the greater distance being the shorter chord.

7. If two sides of one triangle are equal to two sides of another triangle but the angle included between the two sides in the first is greater than the angle included by the corresponding sides in the second, then the third side of the first triangle is greater than the third side of the second.
8. If two sides of one triangle are equal to two sides of another triangle, the third side of the first triangle being greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second triangle.
9. In the same circle or in equal circles, the arcs subtended by unequal chords are unequal in the same order as the chords; and, conversely, chords subtending unequal arcs are unequal in the same order as the arcs.

D. Constructions

Construct a triangle having given two sides and an angle opposite one of them. Discuss the number of solutions to be obtained for various sizes of the angle and for various lengths of sides and altitude.

VIII. REGULAR POLYGONS AND THE CIRCLE

A. New concepts: regular polygon, inscribed polygon, and circumscribed polygon

B. Theorem to be established by intuition [the circumference of a circle is found from the formula $c = \pi d$]

C. Theorems to be proved

°1. If a circle is divided into equal arcs, the chords subtending the arcs form a regular inscribed polygon, and the tangents drawn at the points of division form a regular circumscribed polygon.

2. The area of a regular inscribed polygon is the product of one-half the perimeter by the apothem.

3. The area of a circumscribed polygon is the product of one-half the perimeter and the radius.

D. Constructions

1. To inscribe and circumscribe a square.

°2. To inscribe and circumscribe a regular hexagon.

3. To inscribe and circumscribe a regular decagon.

4. To inscribe and circumscribe a regular fifteen-sided polygon.

*5. To circumscribe a circle about a given regular polygon.

6. To inscribe a circle in a given regular polygon.

E. Relations between the side of a regular polygon and the radii of inscribed and circumscribed circles

1. The side of a square expressed in terms of the radius: $a=\frac{r}{2}$, $a=r\sqrt{2}$.
2. The side of the regular hexagon expressed in terms of the radius: $a=\frac{2\sqrt{3}}{3}r$, $a=r$.
3. The side of the equilateral triangle, expressed in terms of the radius: $a=2r\sqrt{3}$, $a=r\sqrt{3}$.

IX. AREAS

A number of formulas for finding areas have been worked out in the earlier stages of geometry by intuition. They are here to be assumed as true. They are also to be used as bases for the development of further formulas.

A. Summary of formulas previously established

1. The area of a rectangle is equal to the product of the base by the altitude.
⊛2. The area of a parallelogram is equal to the product of the base by the altitude.
3. The area of a square is equal to the square of one side.
⊛4. The area of a triangle is equal to one-half the product of the base and altitude.
⊛5. The area of a trapezoid is one-half the product of the sum of the bases by the altitude.
6. The area of a circle is πr^2, where $\pi=3.141$ and r is the length of the radius.

B. Theorems to be established by proof

1. The area of a triangle is equal to the product of one-half the perimeter by the radius of the inscribed circle.
2. The area of a triangle is equal to the product of the three sides divided by four times the radius of the circumscribed circle.
3. The area of a triangle is given by the formula

$$T=\sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semiperimeter and a, b, and c are the lengths of the sides.

4. The area of an equilateral triangle is given by the formula $T=\frac{a^2}{4}\sqrt{3}$, where a is the length of a side.

⊛5. The areas of similar triangles are to each other as the squares of two corresponding sides.

*6. The areas of two similar polygons are to each other as the squares of two corresponding sides.

C. Theorems to be worked out as exercises

1. Parallelograms having equal bases and altitudes are equal.
2. A parallelogram is equal to a rectangle having the same base and altitude as the parallelogram.
3. A triangle is equal to one-half a parallelogram having the same base and altitude.
4. Triangles having equal bases and altitudes are equal.
5. The sum of the squares of two sides of a triangle is equal to twice the square of one-half of the third side increased by twice the squares of the median to the third side.

⊛6. The area of a regular inscribed polygon is equal to the product of one-half of the perimeter and the apothem.

7. The area of the regular circumscribed polygon is equal to the product of one-half the perimeter and the radius.
8. Two parallelograms are to each other as the products of the bases and altitudes.
9. Two parallelograms having equal bases are to each other as the altitudes.
10. Two triangles are to each other as the products of the bases and altitudes.
11. Triangles having equal bases are to each other as the altitudes.
12. Triangles having equal altitudes are to each other as the bases.
13. The areas of two triangles having one angle equal respectively are to each other as the products of the sides including the equal angles.
14. The area of a sector of a circle is equal to one-half the product of the radius and the length of the arc of the sector.

D. Constructions

1. To construct a square equal to a given rectangle.
2. To construct a square equal to the sum of two or more given squares.
3. To construct a square equal to a given triangle.
4. To construct the square root of an integral number.

Summary.—The geometric materials have been organized in three stages. The first two comprise intuitive geometry and lead the pupil gradually through a period of informal reasoning to the third stage which is largely logical geometry.

The content of the first two stages is presented according to three levels, i.e., measurement of length, measurement of area, and measurement of volume. This chapter has presented a detailed outline of the geometric content of each of the three stages.

Since in the third stage all of the facts previously established by intuition or informal reasoning are assumed, it contains a course briefer than that of traditional demonstrative geometry.

The organization presented in this chapter is intended as a suggestion to teachers and supervisors. It should be supplemented according to their best judgment. Omissions should be made when desirable. In all cases it should be modified and adjusted to the needs of the pupils.

Correlation of two- and three-dimensional geometry.—The outline of the intuitive geometry has shown many contacts between two- and three-dimensional figures. Training in space perception should be derived from both. It is very desirable that such training be continued during the demonstrative geometry usually taught in the tenth school year. For the convenience of those who are interested in the correlation of plane and solid geometry during that stage, the following outline is presented. It lists certain theorems of plane geometry to which others may be added if desired. To the right of each theorem are theorems taken from three-dimensional geometry which may be discussed in some cases and proved in others.

Two-Dimensional Theorems	Three-Dimensional Theorems
1. Through a fixed point in a plane any number of straight lines may be drawn.	Through a fixed point in space any number of lines may be drawn. Through a straight line any number of planes may be passed.
2. Two straight lines intersect in a point.	Two planes intersect in a straight line.
3. If three or more parallel lines are cut by two transversals, the corresponding segments of the transversals are proportional.	If three or more parallel planes are cut by two transversals, the corresponding segments of the transversals are proportional.
4. Two oblique lines drawn from a point to a line which have equal projections are equal.	Two oblique lines drawn from a point to a plane which have equal projections are equal.

5.	Two lines perpendicular to the same line are parallel.	Two lines perpendicular to the same plane are parallel. Two planes perpendicular to the same line are parallel.
6.	Through a point on a given line one line can be drawn perpendicular to the given line.	Through a point on a given line only one plane can be drawn perpendicular to the given line.
7.	From a point outside of a given line only one line can be drawn perpendicular to the given line.	From a point outside of a given line only one plane can be drawn perpendicular to the given line.
8.	Two lines parallel to a third line in the same plane are parallel to each other.	Two lines parallel to a third line in space are parallel to each other.
9.	Two angles in the same plane whose sides are parallel are either equal or supplementary.	Two angles in space whose sides are parallel are either equal or supplementary.
10.	If two parallel lines are cut by a transversal, the alternate interior angles are equal.	If two parallel planes are cut by a third plane, the alternate interior dihedral angles are equal.
11.	A line perpendicular to the radius of a circle at the outer extremity is tangent to the circle.	A plane perpendicular to the radius of a sphere at the outer extremity is tangent to the sphere.
12.	The sum of the angles of a triangle is 180°.	The sum of the angles of a spherical triangle is equal to the spherical excess.

Outline of a course in solid geometry.—The geometry of three dimensions which people in general should know should be taught in the junior high school. The study may be further extended during the course in tenth-grade geometry where it should be related to certain theorems of two-dimensional geometry. In addition, many schools offer a semester course in solid geometry in the eleventh or twelfth school year. Such a course is intended for the pupils who elect it because they are interested or because they need it to satisfy college entrance requirements.

Textbooks offering courses in solid geometry vary little in organization. Most of them begin with an introductory chapter on lines and planes in three-dimensional space. This is followed by one or two chapters on cylinders, cones, and polyhedrons. The last chapter is usually devoted to the study of the sphere. One criticism of this organization is that the solids are studied separately, which is the cause of much overlapping and repetition in the course without any corresponding gain to the learner.

Several years ago some rather extensive experimentation with various arrangements was carried on in the University of Chicago High School. The study extended over a number of years and the outcome was an organization an outline of which is to be presented in the following pages.[8]

LINES AND PLANES IN SPACE

The purpose of this division is to develop the basic concepts for the course and to present many reslationhips between lines and planes in space.

A. Concepts to be developed

Plane	Plane angle
Projection of a point	Parallel planes
Projection of a segment	Line parallel to a plane
Dihedral angle	Perpendicular planes
Measure of dihedral angle	Line perpendicular to a plane

B. Fundamental principles to be discussed informally
1. A line passing through two points in a plane lies wholly in the plane.
2. An unlimited number of planes can be passed through a point.
3. An unlimited number of planes can be passed through a straight line.
4. A straight line and a point not on the line determinê a plane.
5. Three points not in the same straight line determine a plane.
6. Two intersecting straight lines determine a plane.
7. Two parallel straight lines determine a plane.

C. Theorems to be proved
 I. Intersecting planes
  1. If two planes intersect, the intersection is a straight line.
 II. Parallel planes
  1. If two parallel planes are cut by a third plane, the intersections are parallel.

[8] For a detailed discussion of the course see Breslich, *Solid Geometry* (University of Chicago Press, 1929).

2. Parallel segments intercepted by parallel planes are equal.
3. If three or more parallel planes are cut by transversals, the corresponding segments of the transversals are proportional.
4. Two planes perpendicular to the same line are parallel.
5. If two lines are parallel, a plane containing one of them and not the other is parallel to the other.
6. If two angles not in the same plane have their sides parallel and running in the same direction, the angles are equal and their planes are parallel.

III. Lines perpendicular to a plane
1. If one or two parallel lines is perpendicular to a plane, the other is also.
2. If a line is perpendicular to each of two intersecting lines, it is perpendicular to the plane determined by the lines.
3. All perpendiculars to a given line at a given point lie in a plane perpendicular to the given line at that point.
4. Only one line can be constructed perpendicular to a given plane at a given point in the plane, or from a given point outside of the plane.
5. Lines perpendicular to the same plane are parallel.
6. Two lines parallel to the same line are parallel.

IV. Lines parallel to a plane
1. If two lines are parallel, a plane containing one of them and not the other is parallel to the other.
2. If two intersecting lines are parallel to a given plane, their plane is parallel to the given plane.

V. Projections
1. The projection upon a plane of a straight line not perpendicular to the plane is a straight line.
2. Oblique lines drawn from a point to a plane, meeting the planes at points equidistant from the foot of the perpendicular, are equal.
3. Equal oblique lines drawn from a point to a plane meet the plane at points equidistant from the foot of the perpendicular.
4. The acute angle formed by a given line and its projection upon a plane is smaller than the angle it makes with any other line in the plane passing through the point of intersection of the given line and the plane.

VI. Dihedral angles
1. All plane angles of a dihedral angle are equal.
2. Two dihedral angles are equal if their plane angles are equal, and conversely.

VII. Perpendicular planes
1. If a line is perpendicular to a plane, every plane through this line is perpendicular to the plane.
2. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to the intersection is perpendicular to the other.
3. If a plane is perpendicular to two intersecting planes, it is perpendicular to the line of intersection.

VIII. Loci
1. The locus of points in space equidistant from the given points is the plane bisecting the segment joining these points, and perpendicular to it.
2. The locus of points within a dihedral angle and equidistant from the faces is the plane bisecting the dihedral angle.

D. Constructions
1. Through a given point on a given line pass a plane perpendicular to the given line.
2. From a given point outside of a given line, one and only one plane can be constructed perpendicular to the line.
3. At a given point in a plane construct a line perpendicular to the plane.
4. From a point outside of a plane construct a line perpendicular to the plane.
5. Through a line not perpendicular to a given plane construct a plane perpendicular to the given plane.

SURFACES AND SECTIONS OF SURFACES

This part of the course is so organized that the pupil is made acquainted with all of the common solids: the polyhedron, prism, cylinder, pyramid, cone, and sphere. As soon as the basic vocabulary has been taught and the meanings of the new terms have been explained, the theorems relating to sections are presented.

A. Concepts

Polyhedron
Tetrahedron, hexahedron, octahedron, dodecahedron, icosahedron
Face, edge, vertex, surface of a polyhedron
Parallelopiped
Prismatic surface, prism
Slant height
Lateral edge
Base
Altitude
Section, right section
Circle, ellipse, parabola, hyperbola
Surface of revolution

Right and oblique prism
Triangular, quadrangular, etc., prism
Truncated prism
Cylindrical surface, cylinder
Right cylinder, oblique cylinder
Pyramidal and conical surface
Triangular, quadrangular, pentagonal pyramid
Regular pyramid
Frustum of a pyramid
Circular cone, right circular cone
Frustum of a cone
Directrix, generatrix
Cylinder of revolution
Cone of revolution
Similar cylinders, similar cones
Sphere
Center
Radius
Diameter
Section of a sphere
Great circle
Small circle
Poles
Axis of a circle
Tangent line
Tangent plane

B. Theorems to be proved

I. Prisms

1. The lateral edges of a prism are equal.
2. The lateral faces of a prism are parallelograms.
3. The sections of a prism made by parallel planes are congruent.
4. The right sections of a prism are congruent.
5. A section of a prism parallel to the base is congruent to the base.

II. Cylinders

1. The section of a cylinder made by a plane passing through an element is a parallelogram.
2. The sections of a cylinder made by parallel planes cutting all elements are congruent.
3. The sections of a cylinder parallel to the bases are congruent to the base.

III. Pyramids

1. The lateral edges of a regular pyramid are equal.
2. The lateral faces of a regular pyramid are congruent isosceles triangles.

IV. Cones

1. If a pyramid is cut by a plane parallel to the base, the edges and altitude are divided proportionally; the section is a polygon similar to the base; and the areas of the section and the base are proportional to the squares of the distances from the vertex.
2. A section of a cone made by a plane passing through the vertex is a triangle.
3. A section of a circular cone made by a plane parallel to the base is a circle.

V. Surfaces of revolution
1. The lateral areas, or the total areas of similar circular cylinders, or cones, are proportional to the squares of the altitudes, or to the squares of the radii of the bases.
2. The section of a sphere made by a plane is a circle.

AREAS OF SURFACES

It is aimed to develop the formulas for finding lateral and total areas of the various solids. Ample practice is then given in the use of the formulas in practical problems.

A. Concepts

Lateral area	Similar cones of revolution
Total area	Sphere
Surface of revolution	Zone of one base
Similar cylinders of revolution	Zone of two bases

B. Theorems to be proved
1. The lateral areas, or the total areas, of two similar right circular cylinders are proportional to the squares of the altitudes, or to the squares of the radii of the bases.
2. If half of a regular polygon having an even number of sides is revolved about a diagonal joining two opposite vertices, the area of the surface generated is equal to the product of the diagonal by the circumference of the circle inscribed in the polygon.
3. A plane perpendicular to the radius of a sphere at the outer extremity is tangent to the sphere.
4. A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.
5. The intersection of two spheres is a circle whose plane is perpendicular to the line of centers of the spheres and whose center is in that line.

C. Formulas
1. Lateral area

Of a prism, $L=pe$.

Of a right prism, $L=ph$.

Of a regular pyramid, $L=\frac{1}{2}sp$.

Of a frustum of a pyramid, $L=\frac{1}{2}(p_1+p_2)s$.

Of a right cylinder, $L=2\pi rh$.

Of a right circular cone, $L=\pi rs$.

Of a frustum of a right circular cone, $L=\pi(r_1+r_2)s$.

2. Total area
   Of a right circular cylinder, $T=2\pi r(h+r)$.
   Of a cone of revolution, $T=\pi r(s+r)$.
3. Areas of surfaces of revolution.
   Surfaces formed by revolving half of a regular polygon about a diagonal joining two directly opposite vertices, $L=2\pi(MO)(AF)$,
   Surface of a sphere, $S=4\pi r^2$,
   Surface of a zone, $Z=2\pi rh$.

VOLUMES

Formulas are to be developed for finding the volumes of all of the common solids. They are then applied to problems.

A. New terms

Unit of volume, volume	Tangent plane
Inscribed prism, pyramid, and frustum of a pyramid	Spherical segment, spherical cone, and spherical sector
Circumscribed prism, pyramid, and frustum of a pyramid	

B. Facts and principles to be proved

I. Theorems relating to prisms

1. The plane passed through two diagonally opposite edges of a right parallelopiped divides the parallelopiped into two equal triangular right prisms.
2. An oblique prism is equal to a right prism whose base is equal to a right section of the oblique prism and whose altitude is equal to the lateral edge of the oblique prism.
3. The plane passed through two diagonally opposite edges of any parallelopiped divides the parallelopiped into two equal triangular prisms.
4. Prisms having equal bases and altitudes are equal.
5. The volume of a parallelopiped is equal to the product of the area of the base by the altitude.
6. The volume of a triangular prism is equal to the product of the area of the base by the altitude.
7. The volume of any prism is equal to the product of the area of the base by the altitude.

II. Theorems relating to cylinders

1. The volume of a circular cylinder is equal to the product of the area of the base by the altitude.

2. The volumes of two similar cylinders of revolution are to each other as the cubes of the altitudes, or as the cubes of the radii of the bases.

III. Theorems relating to pyramids
1. If two triangular pyramids have equal bases and altitudes, they are equal.
2. The volume of a triangular pyramid is equal to one-third of the product of the area of the base by the altitude.
3. The volume of any pyramid is equal to one-third the product of the area of the base by the altitude.
4. If two solids lie between two given parallel planes, having their bases in these planes, and if the sections made by any plane parallel to the given planes are equal, then the volumes of the solids are equal.

IV. Theorems relating to cones
1. The volume of a circular cone is equal to one-third the product of the area of the base by the altitude.
2. The volumes of two similar cones are to each other as the cubes of the corresponding altitudes.

V. Theorem relating to the sphere

The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

C. Formulas for finding volumes

Rectangular parallelopiped, $v=abc$ and $v=bh$.
Cube, $v=e^3$.
Triangular right prism, $v=bh$.
Triangular prism, $v=bh$.
Right parallelopiped, $v=bh$.
Oblique parallelopiped, $v=bh$.
Prism, $v=bh$.
Cylinder, $v=bh$.
Cylinder of revolution, $v=\pi r^2h$.
Pyramid, $v=\frac{1}{3}bh$.
Frustum of a pyramid, $v=\frac{1}{3}h(b_1+b_2+\sqrt{b_1b_2})$.
Cone, $v=\frac{1}{3}hb$.
Frustum of a cone of revolution, $v=\frac{1}{3}\pi h(r_1^2+r_2^2+r_1r_2)$.
Sphere, $v=\frac{4}{3}\pi r^3$.
Spherical segment of one base, $v=\frac{1}{3}\pi h^2(3r-h)$.
Spherical segment of two bases, $v=\frac{h}{2}(\pi r_1^2+\pi r_2^2)+\frac{\pi h^3}{6}$.
Spherical cone, $v=\frac{2}{3}\pi r^2h$.
Spherical sector, $v=\frac{2}{3}\pi r^2h$.

POLYHEDRAL ANGLES; SPHERICAL POLYGONS

On account of the close relationship between them and also because of the difficulty of the content, these two concepts are discussed together and placed near the end of the course.

A. New terms

Polyhedral angle, trihedral angle	Polar spherical triangles
Similar polyhedrons	Spherical excess
Spherical angle, spherical distance	Congruent and symmetric figures
Spherical polygon	Lune
Spherical triangle	Spherical degree
Right, birectangular, and trirectangular spherical triangles	Quadrant

B. Theorems to be studied

I. Theorems relating to polyhedral angles

1. The sum of two face angles of a trihedral angle is greater than the third.
2. The sum of the face angles of a convex polyhedral angle is less than four right angles.
3. If two trihedral angles have the corresponding face angles equal, the corresponding dihedral angles are equal.
4. *a*) If the face angles of one trihedral angle are equal respectively to the face angles of another,
   b) If two face angles and the included dihedral angle are equal respectively to the corresponding parts of the other,
   c) If two dihedral angles and the included face angle are equal respectively to the corresponding parts of the other,
   d) If the dihedral angles of one are equal respectively to the corresponding parts of the other,
   . Then the trihedral angles are congruent if the parts are arranged in the same order, and symmetric if they are arranged in the reverse order.

II. Theorems relating to polyhedrons

1. The sum of the sides of a convex spherical polygon is less than 360°.
2. Two tetrahedrons having a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the edges including the equal trihedral angles.
3. Two similar polyhedrons are to each other as the cubes of two corresponding edges.
4. There cannot be more than five kinds of regular polyhedrons.

III. Theorems relating to spherical angles and areas
1. All points on a circle of a sphere are equidistant from the poles.
2. If a point on a sphere is at the distance of a quadrant from two given points on the sphere, it is a pole of the great circle passing through the given points.
3. The shortest line that can be drawn between two given points on the surface of a sphere is the minor arc of the great circle which passes through the two points.
4. A spherical angle is measured by the arc of a great circle having the vertex as a pole, and included between the sides produced if necessary.
5. A spherical angle is equal to the dihedral angle formed by the planes of the sides.

IV. Theorems related to spherical polygons
1. The sum of two sides of a spherical triangle is greater than the third side.
2. The sum of the angles of a spherical triangle is less than six and greater than two right angles.
3. If a spherical triangle is the polar triangle of another, then the second is the polar triangle of the first.
4. In two polar spherical triangles each angle of the one is the supplement of that side of the other of which it is the pole.
5. Two spherical triangles are congruent, or symmetric, if they have the following corresponding parts equal: (*a*) three sides, (*b*) three angles, (*c*) two sides and the included angle, or (*d*) two angles and the included side.
6. If two spherical triangles on the same or equal spheres are mutually equiangular they are mutually equilateral, and conversely.
7. The base angles of an isosceles spherical triangle are equal, and conversely.
8. If two angles of a spherical triangle are unequal, the sides opposite are unequal, and conversely.
9. The diameters of a sphere drawn through the vertices of a spherical triangle meet the surface of the sphere in points which are the vertices of a triangle symmetrical to the given triangle.
10. Two symmetric spherical triangles are equal.

C. Constructions

The following constructions are taught:
1. To construct a sphere passing through four given points not all in the same plane.

2. To inscribe a sphere in a given tetrahedron.
3. To determine the diameter of a given material sphere.

D. Formulas

The following formulas are to be proved:

I. The area of a lune is found from the formula $L=2A$, where A is the number of degrees in the angle of the lune and L is the number of spherical degrees.

II. The area of a spherical triangle is found from the formula $T=(A+B+C-180)$, where A, B, and C are the number of degrees in the angles of the triangle and T is the number of spherical degrees. This formula may be stated: $T=E$, the spherical excess.

III. The area of a spherical polygon is found from the formula $P=[s-(n-2)180]$, where s is the number of degrees in the sum of the angles of the polygon and P is the number of spherical degrees. This formula may be stated: $P=E$, the spherical excess.

BIBLIOGRAPHY

Allen, Fiske. "Mathematics in the Senior High School," *School Science and Mathematics*, XXIV (October, 1924), 716–26.

Austin, William A. "A Course in Solid Geometry," *Mathematics Teacher*, XIX (October, 1926), 349–61.

Baker, H. F. "Can the Range of Geometry Taught in School Be Widened?" *Mathematical Gazette*, XIII (May, 1927), 349–57.

Beatley, Ralph. "Notes on the First Year of Demonstrative Geometry in Secondary Schools," *Mathematics Teacher*, XXIV (April, 1931), 213–22.

Betz, William. "The Development of Mathematics in the Junior High School," *First Yearbook: National Council of Teachers of Mathematics* (1926), pp. 141–65.

Evans, George W. "Proposed Syllabus in Plane and Solid Geometry," *Mathematics Teacher*, XXIII (February, 1930), 87–94.

Gugle, Marie. "Geometry in the Junior High School," *ibid.*, pp. 209–27.

Hart, H. F. "Another Phase of the Geometry Situation," *ibid.*, XXIV (March, 1931), 176–77.

Hart, W. W. "Junior High School Mathematics," *School Science and Mathematics*, XXIX (March, 1929), 286–94.

Nunn, T. P. "The Sequence of Theorems in School Geometry," *Mathematics Teacher*, XVIII (October, 1925), 321–32.

Orleans, Joseph B. "The Fusion of Plane and Solid Geometry," *ibid.*, XXIV (March, 1931), 151–59.

Reeve, William D. "The Mathematics of the Senior High School," *Teachers College Record*, XXVIII (December, 1926), 374–86.

Stone, John C. "A One-Year Course in Plane and Solid Geometry," *Mathematics Teacher*, XXIII (April, 1930), 236–42.

Sykes, Mabel. "Some Pedagogical Aspects of Geometry Teaching," *ibid.*, XX (December, 1927), 466–72.

Webb, Harrison E. "The Future of Secondary Instruction in Geometry," *ibid.*, XIV (October, 1921), 337–41.

CHAPTER VIII

THE ORGANIZATION OF THE CONTENT OF ALGEBRA

Three stages in the development of algebra.—The beginning of algebra is as ancient as that of geometry. At the time of Ahmes (1700 B.C.) it had developed far enough to enable the mathematicians to solve verbal equations of the first degree in one unknown quantity. The equations were written as complete sentences. No symbols or abbreviations were employed. The method of solution was that of estimating the answer and correcting errors. The Egyptians were unable to proceed farther because they lacked a sufficiently simple symbolism.

Development of algebra was very slow and occasionally it seemed to stand completely still. It took several thousand years to bring algebra to the final form in which it is studied today. Not until three hundred to five hundred years after Euclid had completed his work on geometry did the Greeks enter upon the second stage in the development of algebra, in which frequently recurring expressions were abbreviated. How difficult it was for them to take the simple step of passing from words to abbreviations is seen from the fact that, even when abbreviations were used, the words frequently were still written in full. The most important Greek mathematician representing the stage of abbreviated algebra is Diophantus of Alexandria who is said to have died about 330 A.D. He wrote a book dealing almost entirely with algebra. Although he used abbreviations for numbers freely and a single symbol for the unknown, he usually described operations in words. Further evidence of the difficulty of passing from the stage of abbreviations to that of symbols is found in the fact that the Arabs, who have contributed so much to mathematics, after becoming acquainted with the work of Diophantus were unable to proceed with it and continued to use words. They did not pass beyond the stage of abbreviations until the thirteenth century. The Italian mathematicians remained in the second stage until the middle of the fifteenth century.

Table XXX contains samples selected from a list of illustrations given by Tropfke.[1] It pictures in a vivid manner how mathema-

TABLE XXX

THE DEVELOPMENT OF MATHEMATICAL SYMBOLS

Regiomontanus (John Mueller) (1464)	$(16x^2+2{,}000=680x)$, 16 *census et* 2,000 *aequales* 680 *rebus*
Michael Stifel (1486)	$(6x^2+8x-6)$, $6z+8\mathcal{X}-6$
Nic. Chuquet (1484)	$(12x+4x^2+6x^3)$, $12^1+4^2+6^3$
Luca Paciuolo (1494)	$(x^2+4=5x)$, 1. *ce.* $\tilde{p}$. 4. *se aguagliano a.* 5. *co.*
Hieronimo Cardano (1545)	(x^2+x+1), 1 *ce.* $\tilde{p}$ 1 *co.* $\tilde{p}$ 1
Michael Stifel (1553)	$(x^3+x^2y-yx^2-y^3)$, $1\,ce+1zA-1\mathcal{X}AA-1AAA$
Raphael Bombelli (1572)	$(x+x^2+x^3)$, $\underset{\smile}{1}\,p.\underset{\smile}{2}\,p.\underset{\smile}{3}$
Simon Stevin (1585)	$(2x^3-4x^2+3x)$, 2 ③ − 4 ② + 3 ①
Petrus Ramus (1592)	$(56x^4+95x^2+36)$, $56bq+95q+36$
Joost Buergi (1552–1632)	$(16x^2-20x^4+8x^6)$, $\overset{\text{II}}{16}-\overset{\text{IV}}{20}+\overset{\text{VI}}{8}$
Johannes Kepler (1619)	$(4x^2-x^4)$, $4^{\text{II}}-1^{\text{IV}}$
Albert Girard (1629)	$(x^3=13x+12)$, 1 ③ *esgale à* 13 ① + 12
Thomas Harriot (1631)	(x^3+bx^2+bcx), $aaa+baa+bca$
William Oughtred (1631)	$(a^4+4a^3b+6a^2b^2+4ab^3+b^4)$, $Aqq+4acE+6AqEq+4AEc+Eqq$
Pierre Hérigone (1634)	$(a^2+a^3+a^4)$, $a2+a3+a4$
René Descartes (1637)	Introduces the modern notation
Pierre de Fermat (1638)	$(2a^3-2a^2x-a^2y+axy)$, *Bc. bis*−*Bq.* in *A bis*−*Bq.* in *E*+*B* in *A* in *E*
Franciscus Vieta (edited by Schooten) (1646)	$(x^3+3a^2x=2b^2)$, $A^3+3B^2A=2Z^2$
Isaac Newton (1685)	Writes a division example: $yy-2ay+aa\overline{)y^4-3\frac{1}{2}aayy+3a^3y-\frac{1}{2}a^4}$
Jacob Bernoulli (1690)	$\left(\frac{b^2}{2a}+\frac{b^3}{2\cdot 3a^2}+\frac{b^4}{2\cdot 3\cdot 4a^3}\right)$, $\frac{bb}{2a}+\frac{b_3}{2\text{ in }3aa}+\frac{b_4}{2\text{ in }3\text{ in }4a_3}$

ticians endeavored to create a satisfactory notation. Regiomontanus (1464) still wrote the equation $16x^2+2{,}000=680x$ in this form: 16 *census et* 2,000 *aequales* 680 *rebus.*

[1] Johannes Tropfke, *Geschichte der Elementar Mathematik I* (Leipzig: Von Veit & Co., 1902).

No less difficult than the second step was the third, in which algebraic symbols begin to make their appearances. This period begins at the close of the fifteenth century, i.e., more than a thousand years after Diophantus. It took European mathematicians until the middle of the seventeenth century to assimilate the meaning of symbols of operations, parentheses, literal number, positive and negative number, exponent, and equation. Yet most textbooks on first-year algebra present all of these ideas within the first thirty pages, and the beginner is expected to comprehend them.

Table XXX also shows the slow development of other mathematical symbols. Thus Luca Paciuolo (1494) still used abbreviations of Latin words, $\tilde{p}$ for plus and $\tilde{m}$ for minus, to indicate addition and subtraction. Cardan (1545), and even Bombelli (1572), are still using p and m in place of the plus (+) and minus (−) signs. The earliest book in which plus and minus signs are used is that of John Widman (1489). However, the letters $\tilde{p}$ and $\tilde{m}$ persisted for a long time after Widman's book was published. Robert Recorde, who wrote the first English treatise on algebra (1557), introduced the equal (=) sign to avoid the tedious repetition of the words "is equal to." Thus the modern form of algebraic symbolism dates back only to the fifteenth century, and its perfection was relatively slow. The multiplication sign × was introduced by Oughtred (1631). The multiplication sign · is used for the first time by Leibnitz (1693).

An important algebraic concept which developed very slowly, although the root of it may be traced back very far in history, is that of "general number." Without this idea real development of algebra would have been out of question. The Hindus used an abbreviation for the unknown several centuries before Diophantus. The Greeks (440 B.C.) had the custom of denoting points, lines, and surfaces by letters. Aristotle (322 B.C.) used capital letters to denote changeable numbers and small letters for determinate numbers. Euclid (300 B.C.) denoted general numbers by letters but associated them with geometric magnitudes. Diophantus used a Greek letter for the unknown number very much as we use x today. He was able to perform algebraic operations without reference to geometry. However, by him as by all mathematicians up to Vieta, known magnitudes are always expressed with specific numbers.

Vieta was the first to leave geometric intuition behind and introduced literal magnitudes (1591) on a large scale, using the vowels for unknown magnitudes and consonants for known magnitudes. He took the great step of using letters for arithmetical coefficients, leading to the idea of general number. However, it took mathematicians until the seventeenth and eighteenth centuries to bring algebra to the form in which we know it today. Descartes (1637) introduced the custom of using the first letters of the alphabet to denote constants and the last letters to denote variables.

The generalization of the conception of number to include the negative number was another slow and difficult process. The idea of positive and negative numbers is detected probably for the first time in the work of Diophantus. However, he solved equations only for positive roots. The Hindu mathematician Bhaskara (1150 A.D.) recognized the existence of absolutely negative quantities but rejected the negative roots of equations. Leonardo de Pisa (1225) admitted a negative solution of an equation whenever a commercial meaning, such as "debt," could be assigned to it. Chuquet (1484) accepted negative numbers as satisfactory. Stifel attached to them the meaning "less than zero" and called them "absurd" numbers. Cardan (1539) did not yet grasp the real significance and importance of negative numbers. He called them *numeri ficti* but admitted them as solutions of equations. How far mathematicians were from comprehending the concept of negative number is seen when we find that Vieta (1540–1603) still excluded all negative solutions of equations and that Harriot (1560–1621) tried to "prove" that equations should have no other but positive solutions. Descartes (1637) represented negative numbers graphically and thus placed them on the same bases as positive numbers. He perfected the computations with negative numbers.

The exponential notation is a good illustration of the slow development of algebraic symbolism. Three thousand years before Christ the Babylonians possessed some knowledge of numbers that are squares and cubes. The Pythagoreans in combining arithmetic with geometry introduced the terms "square" and "cube" to denote second and third power in the sense in which they are used today. This was before Euclid's time. However, the step from the third to the fourth and higher powers was a very difficult one to

take because it was impossible to illustrate them geometrically. Diophantus (third or fourth century A.D.) introduced terms denoting fourth, fifth, and sixth powers of the "unknown number." It was not until Vieta in his *Isagoge in artem analyticam*, 1591 introduced letters to represent coefficients of the unknown that the way was paved for a theory of exponents, but he still used words to denote cubes and squares. The modern notation did not take a complete foothold until the eighteenth century. The story of the development of the exponential notation is clearly told in Table XXX. Descartes (1637) himself, who introduced the present exponential notation, continued to use aa for a^2. Likewise Gauss defended the use of aa on the ground that it was clearer than a^2 and that exponential notation for the higher powers was justifiable mainly because it saved space. In one of the first American manuscripts, written by James Dinman (1730), such expressions as a^2x^5 are used, but aa is still very common. Frequently $aaaaa$ is found, which shows how difficult it was to assimilate the exponential notation. Thus there are powers of $x-y$ written out in full, as $xxxxxx-6xxxxxy+15xxxxyy-20xxxyyy \ldots . yyyyyy$.

As seen from the foregoing brief description of the development of algebra, progress was extremely slow. Verbal algebra, although in use two thousand years before Christ, was unable to advance. The next step toward the use of abbreviations in place of words was taken two thousand years later. Over a thousand years more passed before symbolic algebra, as it is known today, was established. Thus, modern algebra is only three to four hundred years old.

The history of algebra should be enlightening to the teacher of the subject. Algebra had its beginning in problem-solving. It served as an aid to those who were developing the technique of problem-solving. Hence the problem might very well be the central theme in the organization of the first work in algebra.

Progress should not be too rapid. Before algebra is offered to the pupil for study as a science, he should pass through an introductory period in which he should gradually acquire an understanding of the fundamental concepts, symbols, and basic principles. This preliminary algebra should receive considerable attention in Grades VII and VIII, and the algebra offered in Grade IX should be modified to continue the work at the point where the preceding grades

left off. By distributing introductory algebra over several grades, teachers will gain more time for the pupil to attain mastery of the symbolism and fundamental concepts of the subject.

How algebra secured a place in the secondary-school curriculum.— It has been shown that algebra, although its beginning is as ancient as that of geometry, reached its present stage much later than geometry. It did not take the form which it has today until the sixteenth century. In 1707 Sir Isaac Newton published his *Arithmetica universalis*, and in 1720 this work on algebra was translated into English.

American algebra had its beginning in the first quarter of the eighteenth century. At first the subject was taught from notebooks prepared by native Americans who had gone to England to study, but gradually manuscripts and textbooks were written by teachers trained in America. One of the earliest manuscripts was prepared by Isaac Greenwood, who was made professor of mathematics at Harvard in 1728. A second manuscript, dated 1730, was produced by James Dinman, librarian of Harvard College. In 1739 a manuscript appeared which had been written by Samuel Langdon, who became president of Harvard College in 1776. After 1730 American presses began to print textbooks on algebra or including sections on algebra.

Table XV shows that at first algebra was offered in the upper classes of the colleges and that it was gradually moved downward until it became a first-year subject. Thus, in 1718 algebra theses appeared among the commencement theses at Yale, and in 1743 algebra was offered as a first-year subject in that institution.

At Harvard mathematical theses included algebraic subjects as early as 1721. Algebra was included as part of instruction in mathematics in 1726. It was offered in the Sophomore year in 1778, in the Freshman year in 1835, and in 1820 it was required for the first time for admission to Harvard. Gradually it was taken over by the preparatory schools and high schools and in time it was brought down to the first high-school year, where it has remained to the present day.

Because of its close relation to arithmetic, algebra either was usually considered as belonging to arithmetic or was the next course offered. Thus geometry took a higher position in the cur-

riculum than algebra, although historically it was the easier of the two subjects and had developed much earlier.

Why traditional algebra is a difficult school subject.—There has been much dissatisfaction with the content and organization of the materials selected for traditional courses in algebra, especially that of the ninth grade. The following criticisms are frequent:

1. Without offering sufficient introductory experiences to prepare the pupil for the study, first-year algebra is presented in its last stage of development.

2. Not enough experiences are provided to develop real understanding of the concepts of algebra. Literal numbers, signed numbers, symbols of operation, exponents, and equations follow one another in rapid succession, although each alone offers a serious difficulty to the learner. Most textbooks introduce all of these concepts within the first twenty-five to thirty pages. Hence no real understanding is attained by the pupil. The result is confusion and dislike of a subject which is in reality simple if the fundamentals are thoroughly mastered.

3. Too much emphasis is given to mechanical performance of complicated algebraic processes. The subject as presented is extremely symbolic, abstract, and unreal. The practice of beginning algebra with the mechanical work is unhistoric and unpedagogical. Too many pupils learn only to juggle symbols without gaining understanding and number insight.

4. Progress in the subject is too rapid. The major difficulties are not sufficiently separated, and the work is usually carried beyond the needs of the learner.

5. Results are unsatisfactory. Pupils do not know how to use algebra in later courses in mathematics and cannot read with understanding the mathematical literature. In solving problems they do not use algebraic methods, but prefer the more clumsy arithmetical methods.

Algebra in the early grades of the secondary school.—In view of the difficulties experienced by ninth-grade pupils in the study of algebra, frequent recommendations that algebra be dropped as a required school subject are not surprising. The subject needs to be better adjusted to the interests and capacities of the pupils who study it.

Whenever in school work quantitative situations arise in which algebra is more helpful than arithmetic and saves the student's time and work, it should be used. Pupils will be interested if they are shown how important and valuable the subject is in the daily life-experiences and activities of people. They will not find it a difficult subject if the foundation is carefully laid. Algebra can be made sufficiently simple to be taken up as early as the seventh and eighth grades. The danger in teaching is in passing too rapidly to the difficult parts of algebra.

One of the major purposes of the early instruction in algebra is to enable pupils to form clear conceptions of the fundamental notions of the subject. This is easily done through the use of problems and formulas. Problems in percentage, interest, and geometric problems lead to simple formulas that are easily explained and that contain relationships of very simple forms. Geometric formulas are especially suitable because they can be given concrete settings, and because they do not involve signed numbers. Textbooks on algebra are full of formulas, often rather complicated, with insufficient explanations of their meanings. A formula which is not fully explained and illustrated to the point where it is understood is of but little value. Indeed, it may add to the difficulty rather than help remove it. Examples of this type are not hard to find. The use of formulas relating to perimeters, areas, and volumes can at best lead only to mechanical manipulation unless sufficient experiences are offered to make the meanings of these concepts clear. Graphic solutions of equations do not illustrate unless the pupil has previously been taught the fundamentals of graphic methods and procedures.

The study of algebra may be divided conveniently into three stages. The first is introductory. It should fall within the seventh and eighth grades. In the second stage algebra begins to be studied as a science. The third is a continuation of the second. In the first stage three distinct levels of progress should be recognized. They are concerned respectively with the study of expressions of the first, second, and third degrees. To this may be added a unit on the development of the meaning of positive and negative numbers.

The first level in the study of algebra.—The algebra taught at this level is developed gradually (Table XXXI). It is introduced as

need for it arises in the pupils' mathematical work. Only algebraic expressions of the first degree occur at this first level. The study of the most basic and important algebraic concepts and procedures is begun but instruction in them will not cease as long as the pupil studies algebra. Thus problem-solving, relationships, generalizations, solution of equations, and the formula are given considerable attention. The simple algebraic processes are the means for obtaining the results, not the ends, to be attained. Understanding is to be attained by using the simplest illustrations and giving little attention to complicated processes. Major difficulties are being separated. Hence signed numbers and exponents are not yet needed; they have not been touched upon or mentioned.

The material offered for the first level is most suitable for Grades VII or VIII.[2] If no instruction in algebra is given in these grades the material should be presented in the ninth grade.[3] Table XXXI does not give the various facts in the order in which they are to be taught. Sequence should be determined to meet the need and ability of the pupils.

Examination of Table XXXI shows that the beginning of the study of algebra may be made very simple. The use of letters comes first but no algebraic expressions of degree higher than the first need appear. Only a few concepts are introduced and they are always presented concretely. Manipulation is related to problems. The processes taught are the means, not the ends. Progress is slow, months being allowed for the assimilation of the given materials. Problem-solving begins early. The technique of solving problems is taught gradually, all problems being of a simple type.

The second level in the study of algebra.—In the second level the study of linear expressions continues. However, an extension is made by bringing in algebraic expressions of the second degree. The concept of exponent is touched upon. Negative numbers are not needed and are therefore not presented. The second level is reached either near the end of Grade VII or at the start of Grade VIII. The content is presented briefly in Table XXXII. For a more complete discussion the reader is referred to the author's *Seventh-Year Mathematics* and *Eighth-Year Mathematics*. If the

[2] See Breslich, *Seventh-Year Mathematics*.

[3] See Breslich, *Senior Mathematics*, Book I.

TABLE XXXI

ALGEBRAIC SUBJECT MATTER FOR THE FIRST LEVEL

Aims To Be Accomplished	Activities
Understanding of the meaning of literal number.	Using letters as symbols for numbers, as in denoting lengths of line segments, sizes of angles, and lengths of radii of circles.
The meaning of ratio of two literal numbers.	Comparing the sides of a triangle with each other. Finding the ratio of corresponding sides of two triangles.
The meaning of unknown number.	Using letters to denote unknown lengths, unknown sizes of angles, and unknown numbers in verbal problems, such as percentage and interest problems.
The meaning of general number.	Using a letter to denote the lengths of a variable line segment, the lengths of all line segments, the sizes of all angles, and the radii of all circles.
The meaning of linear binomial, trinomial, and polynomial.	Denoting the sum or difference of two line segments or angles, the sum of the sides of a triangle or polygon, and the sum of the angles of a triangle.
Ability to substitute and to evaluate algebraic expressions.	Finding perimeters for particular values of the sides. Finding angle sums of particular triangles. Finding circumferences of circles, percentages, interest, and distances. Making tables for graphs. Checking equations.
Understanding the meaning and use of formulas.	Making formulas for the perimeters of scalene, isosceles, and equilateral triangles. Finding perimeters of polygons. Finding the circumferences of circles. Using percentage and interest formulas.

TABLE XXXI—*Continued*

Aims To Be Accomplished	Activities
Ability to think functionally, to represent linear expressions graphically.	Studying relations between the sides of a triangle and the perimeter, complementary angles, supplementary angles, angles of a triangle, radius and circumference of a circle, distance traveled and time, interest and time, and percentage and base.
Ability to solve by the algebraic method problems leading to equations of the form $ax=b$, $ax+b=cx+d$.	Solving problems relating to perimeters, angles, unknown distances, interest, percentage, and discount.
Ability to solve linear equations in one unknown: 1. Equations of the form $ax=b$ 2. Equations of the form $ax+bx=c$, $ax+b=c$, and $ax+b=cx+d$.	Solving problems leading to linear equations. Solving equations: 1. Informally without the use of axioms 2. By addition, subtraction, multiplication, and division without the use of axioms 3. By the use of axioms
Ability to combine similar terms and to find algebraic sums and differences.	Solving problems and formulas. Practice with abstract exercises.

work is to be done in the ninth school year, helpful suggestions will be found in *Senior Mathematics*, Book I.

As in the first level, symbolic notation, formulas, equations, and manipulative skill should grow gradually out of the solution of problems.

The third level in the study of algebra.—There is to be no abrupt change from one level to the next. Hence the abilities and understandings attained previously are to be further developed. However, the step now to be taken is to pass from functions of the first and second degree to functions of the third. As before, the need for further extension should grow out of problems, especially geometric problems relating to volumes. Computations of surfaces which in-

TABLE XXXII

ORGANIZATION OF ALGEBRAIC SUBJECT MATTER FOR THE SECOND LEVEL

Aims To Be Accomplished	Activities
Understanding of the meaning of new concepts: product of two linear numbers, as bh, and square of a number, as a^2. Meaning of coefficient, of exponent.	Finding areas of rectangles, squares, circles, and other figures.
Ability to generalize.	Representing by single formulas the areas of all rectangles, of all squares, and of all circles.
Appreciation and understanding of formulas.	Finding formulas for the areas of polygons of three or more sides, and of circles. Making formulas for computing percentage, interest, and discount.
Ability to substitute and evaluate.	Finding areas of particular triangles, rectangles, squares, trapezoids, and circles. Finding areas of surfaces that are combinations of the foregoing figures. Finding values of artificial practice exercises containing expressions of the first and second degree.
Understanding of the meaning of parentheses.	Finding areas of trapezoids.
Ability to solve problems.	Solving practical problems in mensuration and percentage, and problems making use of the theorem of Pythagoras.
Proficiency in simple algebraic processes of combining terms, in finding special products of the forms $(a+b)^2$, multiplying linear monomials by linear polynomials, and multiplying linear polynomials by linear polynomials. Ability to extract square root.	Finding areas of surfaces in the form of rectangles and squares. Practice with abstract exercises. Finding the side of a square whose area is known. Finding one side of a right triangle when the other two are given.

TABLE XXXII—*Continued*

Aims To Be Accomplished	Activities
Ability in functional thinking	Studying relations expressed in formulas and in graphs of expressions of the second degree, and in evaluating algebraic expressions.
Ability to solve equations reducing to the form $ax^2=b$.	Finding the side of a square surface, the radius of a circle whose area is given, and one side of a right triangle when the other two are known.

volve expressions and equations of the second degree should be freely interwoven with problems leading to third-degree expressions. Table XXXIII is a brief outline of the content to be presented in the third level of introductory algebra.

TABLE XXXIII

ORGANIZATION OF ALGEBRAIC SUBJECT MATTER FOR THE THIRD LEVEL

Aims To Be Accomplished	Activities
Understanding of such third-degree expressions as a^3, abc, $\frac{prt}{100}$, $\pi r^2 h$, and $\frac{4}{3}\pi r^3$.	Finding the volumes of cubes, rectangular blocks, cylinders, cones, and spheres.
Ability to use formulas of the third degree.	Solving problems leading to formulas of the third degree.
Proficiency in substituting and evaluating.	Computing volumes and areas of surfaces. Practice with abstract exercises.
Proficiency in the processes of algebra.	Performing the processes that arise in the solution of problems.
Ability to solve problems.	Numerous problems are solved which relate to practical applications involving rectangular blocks, cones, prisms, pyramids, cylinders, and spheres. Problems of industry and business are also offered for solution.
Ability to solve equations and to use formulas.	Solving the equations derived from problems. Practice exercises in simple equations.

The meaning of signed numbers.—The history of the development of signed numbers illustrates the difficulty experienced in passing from arithmetic and literal numbers to signed numbers. To begin algebra with the study of signed numbers is to burden the pupil unnecessarily with more than one major difficulty. No negative numbers have been needed in the work outlined so far. However, since signed numbers are used in some of the activities of everyday life the pupil should be made acquainted with them. If the work of the first three levels has been finished, negative numbers should be introduced at the close of the introductory stage or at the beginning of the next stage, i.e., either at the end of the eighth grade or near the beginning of the ninth. The meaning of signed numbers may be established through a study of situations in which they actually arise. Signed numbers were not clearly understood, even by mathematicians, until they were made concrete by means of the scale of numbers. This seems to be still the most convenient and the most intelligible device.[4] However, it should not be the only device used. Other illustrations which appeal to pupils are the graphs of positive and negative numbers, e.g., temperature graphs; the thermometer scale; distances taken in opposite directions, e.g., distances above and below water level; directed forces; gain and loss; and other business uses.

The foregoing outline finishes the introductory stage of algebra. It contains the type of algebra that people in general should know and understand. It might be regarded as the algebra that all pupils should be required to study. If interest has been aroused and if the subject is understood, pupils will be glad to elect the course which is to follow.

The second stage in the study of algebra.—In the first stage the pupil has acquired the understandings and abilities which prepare him for an intensive study of algebra. It is advisable to continue to make problem-solving the outstanding feature of the work. Thus, there should be a unit on problems leading to simple equations in one unknown, one on problems leading to equations in two unknowns, and one on problems leading to quadratic equations. The three units should be preceded by a unit on the operations with signed numbers. Finally, there should be several units on the

[4] See *Eighth-Year Mathematics*, chap. x.

formal processes of algebra with integral expressions and with fractions. It is desirable to offer the formal work as late in the course as possible in order that the pupil may bring to it understanding and appreciation. Such an arrangement is contrary to the traditional organization in which the formal processes are taught as near the beginning of the course as possible.

I. ACQUISITION OF THE ABILITY TO PERFORM THE OPERATIONS WITH POSITIVE AND NEGATIVE NUMBERS

When the pupil has acquired an understanding of the meaning of signed numbers, he is prepared to take the next step, to learn to operate with them.[5] Emphasis should be at first on simplicity of exercises and on understanding rather than on mechanical performance. The aim is a mastery of the laws of signs which will supply the pupils' needs in solving the equations that occur in the next three units. The processes with polynomials may be taught later.

The following is an outline of the content of the unit on signed numbers.

A. Adding positive and negative numbers
   1. Adding graphically two or more terms
   2. Interpreting addition of signed numbers by concrete illustrations other than graphs, such as gain and loss, or forces acting in the same or opposite directions
   3. Deriving the law for adding monomials; practicing the use of the law
   4. Combining similar terms and using the law of order in addition

B. Subtracting positive and negative numbers
   1. Subtracting numbers in the number scale; subtracting by means of the thermometer scale and other devices; deriving the law of signs in subtraction; practicing the use of the law
   2. Combining similar terms

C. Multiplying positive and negative numbers
   1. Deriving the law of signs for multiplying signed numbers
   2. Multiplying by zero
   3. Practice in multiplying monomials

D. Dividing positive and negative numbers
   1. Deriving the law of signs in division
   2. Practice in dividing monomials

[5] *Senior Mathematics*, Book I, chap. iv.

II. DEVELOPING THE ABILITY TO SOLVE PROBLEMS LEADING TO LINEAR EQUATIONS IN ONE UNKNOWN

Most of the preliminary work preparatory to a unit in problem-solving should be done before the pupils come to this unit. Thus they should have acquired considerable knowledge about the methods of solving problems and about deriving and solving equations. This knowledge is now to be systematized and rounded out.[6] The variety of problem material should be increased. The axioms should be used in the solutions of equations. The following should be the outcomes of the unit on problem-solving:

A. Facility in solving verbal problems. (To attain this objective the teacher should use a large number of problems and of problem situations. The technique of problem-solving should receive emphasis throughout the unit.)
B. Facility in deriving the equations for solving verbal problems. (Special training in deriving equations should be offered.)
C. Mastery of the simple processes which occur in the solutions of equations
D. Proficiency in solving equations
   1. Equations reducing the form $ax=b$ when a and b are common or decimal fractions. (The division axiom should be used in solving $ax=b$.)
   2. Equations reducing to the form $ax \pm b=c$. (The addition and subtraction axioms are to be used.)
   3. Equations of the forms $\frac{x}{a}=c$ and $\frac{x}{a} \pm \frac{x}{b}=c$. (The multiplication axiom should be used.)
   4. Equations reducing to the form $ax+b=cx+d$, where a, b, c, and d are common fractions or decimal fractions, positive or negative. (Practice is offered in the use of all axioms and in the processes entering into the solutions.)
   5. Simple equations involving parentheses
   6. Simple formulas, to be solved for one of the variables

III. ACQUIRING ABILITY TO SOLVE PROBLEMS LEADING TO LINEAR EQUATIONS IN TWO UNKNOWNS

Unit III continues the work of Unit II. Problems motivate the manipulative work which is still very simple. Solutions of equa-

[6] *Ninth-Year Mathematics*, chap. iii.

tions in three or more unknowns should be optional. The unit objectives are:

A. Understanding of the graphical method of solving simultaneous equations. (The graphical method is not as practical as some of the other methods of solving simultaneous equations. However, it is illuminating and instructive. It aids in understanding the meaning of a solution of a system of equations. It gives practice in graphical work and serves as a motive for the further study of graphs.)

B. Proficiency in the use of algebraic methods. (Several methods of eliminating variables should be taught. It may then be left to the pupil to choose the method he considers the most effective. Checking the solutions gives practice in the use of the processes.)

IV. ACQUIRING ABILITY TO SOLVE PROBLEMS LEADING TO QUADRATIC EQUATIONS

No scientific study of the quadratic equation should be attempted. That type of work belongs to a later stage. In this unit it is intended to acquaint the pupil with a method of solving the type of quadratics that he is likely to meet in his other school subjects. Practice in the fundamentals, familiarity with the graphs, and skill in problem-solving are to be given much attention.[7] The following methods and processes should be learned:

A. Graphical solution of quadratic equations. (The method is taught not only because it enables the pupils to solve equations, but also because it throws light on certain difficulties encountered by pupils in the study of quadratic functions and equations. It offers practice in substitution, evaluation, and the simple processes of algebra. From the standpoint of mathematical training it is a valuable method to teach.)

B. Algebraic solution of quadratic equations. (Since a general method is needed, which will solve any given quadratic equation, the "completing of the square" method should always be taught. The method is also valuable because it gives good practice in the fundamental processes. The factoring method is not a general method of solution and need not receive emphasis except when practice in factoring is desired.)

C. Sufficient practice should be given in simplifying square roots of arithmetical numbers, to help the pupil simplify the solutions of equations found by the "completing of the square" method.

[7] *Ibid.*, chap. vii; *Senior Mathematics*, Book I, chap. xi.

V. DEVELOPMENT OF SKILL IN THE FOUR FUNDAMENTAL PROCESSES OF ALGEBRA

This unit is a review, summary, and extension of what has been previously taught about the fundamental processes.[8] It is intended for all pupils who expect to continue the study of algebra and mathematics. The character of the work may be made more abstract and mechanical than that of any of the preceding units. The following processes are to be learned:

A. Addition and subtraction
  1. Review of the laws of signs in addition and subtraction of monomials
  2. Addition and subtraction of polynomials
     a) Horizontal addition and subtraction
     b) Vertical addition and subtraction
     c) The use of parentheses in indicated additions and subtractions of polynomials

B. Multiplication
  1. Review of the law of signs in multiplication; multiplication by zero
  2. Laws of exponents in multiplication
  3. Multiplication of monomials
  4. Multiplication of polynomials by monomials
  5. Multiplication of polynomials

C. Division
  1. Review of the law of signs in division
  2. The law of exponents in division
  3. Division of monomials by monomials
  4. Finding monomial factors
  5. Reduction of quotients
  6. Long division.

VI. ACQUIRING KNOWLEDGE OF SPECIAL PRODUCTS AND THEIR FACTORS

It is economical to teach the first special products and factoring together.[9] The following products should be learned in this unit:

1. $a(b+c)=ab+ac$
2. $(a+b)^2=a^2+2ab+b^2$
3. $(a-b)^2=a^2-2ab+b^2$
4. $(a+b)(a-b)=a^2-b^2$
5. $(ax+b)(cx+d)=acx^2+(ad+ac)x+bd$

[8] *Senior Mathematics*, Book I, chap. xii.

[9] *Ibid.*, chap. xiii.

Factoring should be taught as the inverse of multiplication. The pupil should learn to factor the following by inspection.

1. Polynomials containing a factor common to all terms
2. The difference of two squares
3. The quadratic trinomial square
4. The general quadratic trinomial of the form ax^2+bx+c

VII. ATTAINING AN UNDERSTANDING OF THE FUNDAMENTAL OPERATIONS WITH FRACTIONS

VI and VII may be merged.[10] The operations with fractions may form the basis of organization and factoring may be taught as it is needed in the simplification of the exercises. The combination motivates factoring because it is needed in the various processes with fractions. The unit should develop the following five abilities in the order indicated by the numerals:

1. To change a given fraction to the simplest form
2. To multiply fractions
3. To divide by a fraction
4. To add and subtract fractions
5. To perform the foregoing processes when they occur jointly in problems and exercises

The third stage in the study of algebra.—If the pupil has studied introductory algebra in Grades VII and VIII he should complete the seven units of the second stage as outlined in the foregoing pages by the time he finishes the ninth grade. The work of the third stage usually falls into the period of the eleventh grade, but some schools follow the practice of offering it in the tenth. The content of the course is fairly well standardized, but the organization of the materials varies. When the study of algebra is discontinued during the tenth grade, teachers usually begin the third stage with a review of the laws and processes taught in the previous courses. A review seems to be necessary with most classes. However, it has been found uneconomical to let the review be merely a going-over of the old work in the same way. There is much to be gained by presenting the old in a new setting.

Experimentation in the laboratory school of the University of Chicago has developed an organization which has given gratifying

[10] *Ninth-Grade Mathematics*, chap. vi.

results.[11] The function concept is made the unifying idea of the course. Functional thinking is stressed. As in the first two stages, verbal problems continue to receive considerable attention. The following topics indicate the content of the course:

I. Linear polynomials and equations
II. Quadratic polynomials and equations
III. Polynomials and equations of degree higher than the second
IV. Fractions and factoring
V. Exponents; irrational equations; radicals
VI. Logarithms
VII. The binomial theorem; progressions
VIII. Systems of quadratic equations in two unknowns

I. ATTAINING UNDERSTANDING OF LINEAR POLYNOMIALS AND EQUATIONS

A. Algebraic concepts to be acquired by the pupils

Algebraic expression, binomial, coefficient, constant, constant of variation, difference, direct variation, elimination, equivalent equations, factor, graph of a linear function, graph of a linear equation, graphical solution, inconsistent equations, linear equation, linear function, literal equation, literal number, monomial, parenthesis, polynomial, product, solution of equation, system of equations, sum, term, trinomial, value of literal number, and variable.

B. Training in solving problems
   1. Learning how to derive the equation from a problem
   2. Solving problems in direct variation

C. Practice in functional thinking obtained in
   1. Evaluation of linear functions
   2. Graphical representation of linear functions

D. Ability to solve equations
   1. Linear equations
      a) Containing parentheses
      b) Containing fractional coefficients.
   2. Fractional equations
   3. Literal equations
   4. Equations in two unknowns
   5. Equations in three unknowns

E. Skill in the use of methods of solving equations
   1. The graphical method
   2. Methods of elimination

[11] *Senior Mathematics*, Book III.

F. Proficiency in algebraic processes
   1. The laws of signs
   2. Addition, subtraction, multiplication, and division of algebraic expressions

II. ATTAINING UNDERSTANDING OF QUADRATIC POLYNOMIALS AND EQUATIONS

A. Concepts to be acquired:

Complex number	Parabola
Hyperbola	Quadratic formula
Imaginary number	Quadratic function
Inverse variation	Rational number
Irrational number	Real number
Normal form of a quadratic equation	Zero of a function

B. Symbols to be understood: $f(x)$, $+\infty$, $-\infty$

C. Training in functional thinking obtained by
   1. Graphical representation
      a) Making graphs of quadratic functions
      b) Discussion of graphs
      c) Finding the zeros of functions
   2. Evaluation of functions
   3. Factoring quadratic functions
      a) Types
         (1) Difference of squares
         (2) Quadratic trinomial
      b) Methods
         (1) Trial method
         (2) Formula method

D. Understanding of the number system
   1. Real numbers
      a) Rational, i.e., positive and negative whole numbers and fractions
      b) Irrational
      c) The real number scale
   2. Imaginary numbers and complex numbers

E. Study of inverse variation including
   1. Graphical representation of the law of variation
   2. Solving problems in variation

F. Developing understanding of the quadratic equation
   1. Equations of the form $ax^2+bx+c=0$
      a) Methods of solving
         (1) Graphical method
         (2) Formula method
         (3) Factoring method

b) Nature of the roots of quadratic equations
c) Relations between the roots and coefficients
2. Equations leading to quadratics
a) Fractional equations
b) Literal equations
c) Formulas

G. Training in problem-solving by using problems
1. Leading to quadratics
2. Leading to fractional equations
3. Leading to literal equations

III. ACQUIRING AND UNDERSTANDING OF POLYNOMIALS OF A DEGREE HIGHER THAN THE SECOND

A. Processes to be learned
Division of polynomials
Synthetic division
Evaluation of polynomials
Extraction of the square root of a polynomial

B. Principles to be understood
Remainder theorem
Factor theorem

C. Equations of degree higher than the second are to be solved by graph, by factoring, and by the method of reduction to quadratic equations

IV. DEVELOPMENT OF PROFICIENCY IN THE OPERATIONS WITH FRACTIONS AND IN FACTORING

A. Fractions
1. The processes with fractions taught in the preceding course are reviewed and extended to include fractions of greater complexity
2. Complex fractions of a simple type are to be studied

B. Factoring is taught in connection with the operations with fractions; it includes the following types:
1. Binomials
a) The difference of two squares
(1) $x^2-y^2=(x+y)(x-y)$
(2) $(a+b)^2-(s+t)^2=(a+b+s+t)(a+b-s-t)$
b) The difference of two cubes
$x^3-y^3=(x-y)(x^2+xy+y^2)$
c) The sum of two cubes
$x^3+y^3=(x+y)(x^2-xy+y^2)$
d) The sum or difference of like powers higher than the fourth
x^5-y^5; x^5+y^5; x^7-y^7; x^7+y^7

2. Trinomials
   a) The trinomial ax^2+bx+c
      Factored by trial
      (1) $10x^2-17x+3=(2x-3)(5x-1)$
      Factored by formula
      (2) $ax^2+bx+c=a(x-x_1)(x-x_2)$, x_1 and x_2 being the roots of the equation $ax^2+bx+c=0$
   b) The trinomial square
      $x^2\pm 2xy+y^2=(x\pm y)^2$
   c) The incomplete trinomial square
      $x^4+x^2y^2+y^4=(x^4+2x^2y^2+y^4)-x^2y^2=(x^2+y^2+xy)(x^2+y^2-xy)$
3. Polynomials, not including the forms given in 1 and 2
   a) Polynomials containing a common monomial factor
      $ax+ay+az=a(x+y+z)$
   b) The perfect cube of a binomial, factored by grouping
      $a^3\pm 3a^2b+3ab^2\pm b^3=(a\pm b)^3$
   c) Polynomials whose terms may be so grouped as to change them to one of the preceding forms
      (1) $x^2+2xy+y^2-k^2=(x+y)^2-k^2$
      (2) $x^2+2xy+y^2-a^2+2ab-b^2=(x+y)^2-(a-b)^2$
      (3) $x^2+2xy+y^2-5x-5y+6=(x+y)^2-5(x+y)+6$
   d) Polynomials containing binomial factors of the form $x\pm a$, as
      $3x^3-x^2-4x+2=(x-1)(3x^2+2x-2)$

V. ATTAINING FACILITY IN THE USE OF EXPONENTS, RADICALS, AND IRRATIONAL EQUATIONS

A. Concepts to be understood

Base	Order of a radical
Conjugate radicals	Power
Exponent	Radical
Fractional exponent	Radicand
Index of a root	Rationalizing a denominator
Irrational equation	Similar radicals
Negative exponent	Zero exponent

B. Ability to use the laws of exponents

1. $a^m \cdot a^n=a^{m+n}$
2. $\dfrac{a^m}{a^n}=a^{m-n}$
3. $(a\cdot b\cdot c)^m=a^mb^mc^m$
4. $\left(\dfrac{a}{b}\right)^m=\dfrac{a^m}{b^m}$
5. $(a^m)^n=a^{mn}$
6. $a^0=1$
7. $a^{-m}=\dfrac{1}{a^m}$
8. $a^{\frac{1}{n}}=\sqrt[n]{a}$
9. $a^{\frac{m}{n}}=\sqrt[n]{a^m}$
   $=(\sqrt[n]{a})^m$

C. Proficiency in the following processes
   1. To change an expression containing negative or zero exponents to an identical expression free from negative or zero exponents
   2. To simplify irrational expressions
   3. To reduce a fractional radicand to the integral form
   4. To reduce the order of a radical
   5. To add and subtract radicals
   6. To multiply radicals of the same or of different orders
   7. To divide by a radical
   8. To rationalize the denominator of a fraction
   9. To find the square root of a binomial of the form $x+a\sqrt{y}$
   10. To solve irrational equations

VI. DEVELOPING FACILITY IN THE USE OF LOGARITHMS

The following facts and processes are to be acquired:

1. The meaning of logarithms
2. Using logarithms to multiply, divide, raise to powers, and extract roots
3. Developing a rule for finding the characteristic of the logarithm of a given number
4. Appreciation of the degree of accuracy to be attained in computations involving data obtained by measurement
5. Ability to use the process of interpolation in finding the logarithms of five-figure numbers and in finding logarithms of the functions of angles given in degrees, minutes, and seconds
6. Ability to solve simple exponential equations by means of logarithms

VII. ATTAINING UNDERSTANDING OF THE BINOMIAL THEOREM AND THE PROGRESSIONS

A. Concepts to be acquired:

Arithmetical means	Elements of a progression
Arithmetical progression	Factorial
Binomial theorem	Geometric means
Common difference	Geometric progression
Common ratio	Infinite progression

B. Processes to be learned
   1. To expand a power of a binomial to a given number of terms
   2. To find a required term in the expansion of a power of a binomial
   3. Given three elements of a progression, to find the remaining two

4. To insert a number of arithmetical, or geometric, means between two given numbers
5. To find the sum of n terms of a progression
6. To find the sum of an infinite geometric progression

C. Formulas to be learned

1. $(a+b)^n=a^n+n(a)^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\cdots\cdot$
2. $t_r=\frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1\cdot 2\cdot 3\ldots\ldots(r-1)}a^{n-r+1}b^{r-1}$
3. $l=a+(n-1)d.$
4. $s=\frac{n}{2}(a+l)$
5. $l=ar^{n-1}$
6. $s=\frac{a(1-r^n)}{1-r}$
7. $s=\frac{a}{1-r}$, if $r<1$ and if the number of terms is unlimited

VIII. ACQUISITION OF THE ABILITY TO SOLVE SYSTEMS OF QUADRATIC EQUATIONS

A. Types of equations

1. The equation $y=ax^2+bx+c$, representing a parabola
2. The equations $x^2+y^2=a^2$ and $x^2+y^2+ax+by+c=0$, representing circles
3. The equations $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, or $b^2x^2+a^2y^2=a^2b^2$, representing ellipses
4. The equations $xy=c$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, representing hyperbolas
5. Irrational and fractional equations in two unknowns

B. Ability to solve systems of equations

The following are examples of the systems of equations to be studied:

1. $\begin{cases} x+y-2=0 \\ 4x^2-2x+5=y \end{cases}$
2. $\begin{cases} x^2+y^2=25 \\ 4x^2+9y^2=144 \end{cases}$
3. $\begin{cases} x^2+2xy+y^2+x+y-12=0 \\ x^2+y^2=36 \end{cases}$
4. $\begin{cases} x^2+y^2=49 \\ xy=24 \end{cases}$
5. $\begin{cases} x^2+y^2+xy=4 \\ x^2+2y^2-xy=7 \end{cases}$
6. $\begin{cases} x^3-y^3=21 \\ x-y=3 \end{cases}$
7. $\begin{cases} x+y-x^2y^2=2 \\ x^2+y^2-x^4y^4=4 \end{cases}$
8. $\begin{cases} \frac{1}{x^3}+\frac{1}{y^3}=72 \\ \frac{1}{x}+\frac{1}{y}=6 \end{cases}$
9. $\begin{cases} \sqrt[3]{x}+\sqrt[3]{y}=3 \\ x+y=27 \end{cases}$

C. Methods of solution to be learned
   1. Graphic method
   2. Algebraic method
      a) Elimination by substitution
      b) Elimination by addition or subtraction
      c) Elimination by division of one equation by the other
      d) Deriving linear equations from a quadratic by factoring
      e) Change of variables

D. Cases of systems to be solved
   1. One linear equation and one quadratic
   2. Both quadratic
      a) In which one of the unknowns may be eliminated by addition or subtraction
      b) In which one equation can be resolved into linear factors
      c) In which one equation is of the form $xy=c$
      d) In which all terms containing the unknowns are of the second degree
   3. Equations of degree higher than the second, one of which is divisible by the other
   4. Irrational and fractional equations, solved by substituting a new variable

The foregoing organization is practically that used in *Senior Mathematics*, Book III. Since most textbooks on eleventh-grade algebra contain ample instructional material that may be used in teaching the various units, the teacher will have no difficulty in finding subject matter to supplement the outlines of the units.

BIBLIOGRAPHY

Birch, J. J. "Algebra in the Junior High School," *Education*, XLVI (September, 1925), 18–27.

Chateauneuf, Amy Olive. *Changes in Content of Elementary Algebra.* Doctor's thesis, University of Pennsylvania, Philadelphia, Pa., 1929.

English, Harry. "College Preparation: What Is Its Effect on What You Teach and How You Teach It?" *Mathematics Teacher*, IX (September, 1916), 2–32.

Gentleman, F. W. "The Content of a Mathematical Course for the Junior High School," *ibid.*, June, 1917, pp. 209–18.

Hinkle, E. C. "Algebra in the Junior High School," *School Science and Mathematics*, XXV (March, 1925), 271–87.

Mirick, Gordon R., and Sanford, Vera. "An Elective Course in Mathematics for the Eleventh and Twelfth School Years," *Mathematics Teacher*, XIX (April, 1926), 235–41.

Nunn, T. P. *The Teaching of Algebra*, pp. 51–56. London and New York: Longmans, Green & Co., 1927.

Reeve, W. D. "Eleventh Year Mathematics Outline," *Mathematics Teacher*, XXIII (November, 1930), 413–37.

Rugg, H. O., and Clark, J. R. *Scientific Method in the Reconstruction of Ninth Grade Mathematics*, "University of Chicago Supplementary Educational Monographs," Vol. II, No. 1.

Schorling, Raleigh, and Clark, John R. "A Program of Investigation and Cooperative Experimentation in the Mathematics of the Seventh, Eighth and Ninth School Years," *Mathematics Teacher*, XIV (May, 1921), 264–75.

Simons, Lao Genevra. *Introduction of Algebra into American Schools in the Eighteenth Century*, U.S. Bureau of Education, Bull. 18 (1924).

Smith, D. E. "A Glimpse of Early Colonial Algebra," *School and Society*, VII (January 5, 1918), 8–12.

CHAPTER IX

THE CORRELATION OF MATHEMATICAL SUBJECTS

Extending the period of secondary-school mathematics to include the seventh and eighth grades.—In the foregoing chapters attention has been called repeatedly to the numerous criticisms of the traditional organization of the mathematical materials. The seriousness and importance of the problem is further reflected in the large amount of discussion devoted to it in the mathematical literature. Far more has been written on questions relating to the curriculum than on questions about the teaching of mathematics.

One of the greatest opportunities for a thorough reorganization of the mathematical curriculum has presented itself in the junior high school movement. It gives promise, if properly taken advantage of, to eliminate the most outstanding faults of the traditional organization. The fact has become more and more evident that the senior high school attempts to do the impossible by crowding all the introductory algebra into the first year and all the plane geometry into the next. In the junior high school it is possible to begin secondary-school mathematics in the seventh grade. It may then be continued without interruption as long as teachers are successful in holding the interest of the pupils.

The plan is not new. European schools have always taught algebra and geometry for a period of years without a break in either subject. In the junior high school movement American schools are offered an opportunity which Europeans have long ago enjoyed and which indeed has given the European schools a decided advantage over the American.

Organizations of algebraic, geometric, and other mathematical materials suitable for instruction in junior high schools have been presented in previous chapters. However, there is no reason why the elementary schools should not introduce in the seventh and eighth grades the first two years of the new junior high school cur-

riculum in mathematics. The junior high school movement has shown that the problem of mathematics is in no small degree one of curriculum construction.

Presenting secondary-school mathematics as separate subjects.—The question arises as to the order in which the mathematical subjects are to be offered for instruction. Writers do not agree on the question. Some insist that the mathematical subjects be kept separated from each other in the manner of the traditional senior high school courses. They would begin with half a year of arithmetic. They would then devote the second half to intuitive geometry. It, in turn, may be followed by a semester's work in algebra. The advocates of this plan do not deny that there are points of contact in the three subjects, but they regard them as essentially different in character. They contend that correlation would be unnatural and therefore detrimental to each.

The plan has been severely attacked by some writers who regard it as unpedagogical because it tends to rush the pupil through each subject to the exclusion of all others. They contend that it develops among the pupils a deplorable attitude of "getting through" each subject and that real assimilation is therefore difficult to attain.

Simultaneous teaching of several mathematical subjects.—A second plan which has been indorsed by various writers is that of teaching the mathematical subjects in parallel courses. Thus, algebra and geometry would be started together and then carried simultaneously for two years instead of each being taught separately for one year. According to the plan, some topic in either subject would be taken up and fully developed. It would then be followed by a topic selected from the other subject. In this manner the subjects will alternate throughout the course. It is claimed by the advocates of the plan that it is conducive to a thorough assimilation because ideas are being presented more slowly than when the subjects are separated; that it is psychological because the simple ideas of both subjects are presented and taught before the more difficult are taken up; that it enables the teacher to use either subject to clarify the other; and that it eliminates the waste of time which arises from the interruption of algebra when the entire year following the course is devoted exclusively to geometry.

Those who have tried the parallel plan report the following

difficulties and objections. Since it is necessary to use two textbooks, one for each subject, pupils are frequently at a loss as to which book they should bring to school. Often they forget whether the next lesson is to be on algebra or geometry. To the learner the frequent change from one subject to the other seems abrupt and uncalled for. The interrelations are not always apparent to him because such relations often do not exist. Changes from one subject to the other occur frequently when interest runs high or when understanding has not yet been attained. In each case an interruption is likely to be disastrous. Taking up a topic after it has been dropped for days, the teacher usually finds it necessary to have a review to establish connection between the new work and the old, which always causes considerable loss of time. Thus the advantages which the plan offers to the pupil are likely to be more than offset by the serious disadvantages.

The direct values derived by those who have tried the parallel plan seem to be in the opportunities for using the relationships existing between the mathematical subjects and in the possibilities of interweaving them. Particularly valuable is the material in which algebra may be used in solving geometric problems. It helps the pupil to manipulate the geometric subject matter, and algebra is thereby kept under constant review.

The search for unifying factors in the teaching of mathematics.— The traditional order of keeping the mathematical subjects separated has been criticized as pedagogically unsound by leaders in mathematics and in education the world over. Ideas of number and space should not be separated because they are naturally closely related. This fact has been readily accepted in such courses as trigonometry, analytic geometry, and calculus. To eliminate sharp lines of demarcation between arithmetic, algebra, and geometry is just as important in the lower courses of secondary-school mathematics. Furthermore, much would be gained by relating mathematics closer to other school subjects and to the applications of everyday life. Pupils will learn mathematics more easily if they are made to feel that the subject is worth the effort required to master it.

Klein[1] of Germany suggested that algebra and geometry be

[1] F. Klein, "Ueber den mathematischen Unterricht der Hoeheren Schulen," *Jahresbericht der Deutschen Mathematiker Vereinigung*, 1902, pp. 128–40.

joined by making the function concept the unifying idea in mathematics. He further recommended that plane and solid geometry be closely related and that a psychological arrangement be insisted on.

Perry of England urged close relationship between the mathematical subjects and the applications in the sciences and in engineering. He advocated that more of the useful parts of mathematics be brought down to the lower courses.

The movement to derive abstract relations of algebra through the more concrete experiences of geometry encouraged close relationships between algebra and geometry.

Tannery and Borél of France recommended a very close connection among the mathematical subjects and advocated the bringing-down into the lower courses of the elements of analytic geometry and calculus.

Charles W. Eliot held that arithmetic, algebra, and geometry be taught together from beginning to the end, each subject illustrating and illuminating the other two.

Judd says:

> The application of algebra to geometry is no less an extension of the intellectual associations of a pupil than is the application of algebra to calculations called for in problems in physics. . . . . The mistake narrowing the pupil's thinking can be avoided by good teaching. The best way to avoid this narrowing of the pupil's intellectual horizon is not to fill textbooks on algebra with artificial forms of thinking about supposedly practical situations, such as geographical distances and ages of imaginary people, but to seek legitimate associations of algebra with geometry and other sciences.[2]

Young[3] recommended simultaneous instruction in algebra and geometry in which the teacher should take advantage of the interrelations of the two subjects.

Moore suggested that algebra, geometry, and physics of the secondary school be organized into a thoroughly coherent course.

It is evident that all of the foregoing recommendations aimed to bring about improvement in the teaching of mathematics by establishing close relationships among the mathematical subjects,

[2] Charles H. Judd, *Psychology of Secondary Education*, p. 143.

[3] J. W. Young, *The Teaching of Mathematics*, pp. 183–85.

between mathematics and other school subjects, and between mathematics and everyday life-experiences.

The unification of the mathematical subjects.—It has been shown that each movement for the improvement of secondary mathematics attempted the solution of the problem by the establishment of some kind of relationship. The movement of unifying the mathematical subjects may therefore be regarded as the natural and logical outcome of former movements.

Moreover, it may be justified in the historical development of mathematics. The Greeks used geometry to clarify abstract algebra by making it concrete. Whenever they found this impossible they were unable to proceed. They could not picture geometrically powers higher than the third. Hence, they did not go beyond the third power. However, they could perform the difficult process of extracting the square root of a number because they found how to do it geometrically. They verified certain fundamental algebraic identities, such as $(a+b)^2=a^2+2ab+b^2$ by diagrams, treated proportions geometrically, and solved quadratic equations by geometric methods. Thus the Greeks found it helpful and necessary to correlate algebra and geometry in the development of mathematics. The practice continued, and until the eighteenth century mathematicians in general made frequent use of geometric figures in algebra.

In American schools algebra and geometry were not always taught as distinct sciences. Examination of the first manuscript written in the early part of the eighteenth century shows that the subjects were correlated and that algebraic methods were used freely in the solution of geometric problems. However, in the organization of the mathematical materials writers were influenced by the plan used in the English universities. They classified subject matter separately as arithmetic, algebra, and geometry. Later the connections between the subjects were gradually lost. At the end of the eighteenth century the separation was definitely accomplished.

In time the courses in arithmetic, algebra, and geometry were moved down bodily from the college into the high school. Difficulties began to appear. It was found that the arrangement originally designed for college students was not suitable for instruction in the

high school. One of the first difficulties which teachers had to overcome was that of bridging the gaps between the various subjects. The transition from arithmetic to algebra and from algebra to geometry was not as simple a matter for high-school pupils as for the college students. It was being realized that the solution of the problem would have to be sought in bringing the separate subjects into closer relation to each other. Accordingly, attempts were made to acquaint the pupil early and gradually with the symbolic notation of algebra and with algebraic methods of solving problems, and to introduce instruction in geometry long before the study of demonstrative geometry. The feeling that the solution of the difficulty may be found in a closer relation of the mathematical subjects was growing, and in 1893 the Committee of Ten of the National Educational Association added its influence by recommending instruction in the grammar school in algebraic symbols and simple equations and systematic instruction in concrete and experimental geometry beginning at the age of ten. It was not presumed that algebra and geometry in the early grades should appear as separate subjects, but that they should be taught in connection with arithmetic. Thus, the way was opened to teachers to experiment with the correlation of the mathematical subjects.

Interest in the problem was greatly stimulated by Professor Moore[4] of Chicago. He frankly recommended the breaking-down of the traditional organization of the mathematical curriculum. He advocated the abolition of "water-tight compartments" by which arithmetic is taught in one compartment, algebra in another, geometry in another, and trigonometry in another. He presented vigorously the contention that mathematics should be taught as one subject, each division helping and illuminating the other.

The correlation of mathematical subjects is full of beneficial possibilities. While each of the former movements aimed to correct certain particular shortcomings of the curriculum, or to remove some of the most objectionable features, it is probable that the plan of correlation and unification may arrive at a real solution of the problem of reconstructing the mathematical curriculum in a satisfactory and thoroughgoing manner.

[4] E. H. Moore, "On the Foundations of Mathematics," *Science*, XVII (March, 1903), 401–16.

Professor Moore's address aroused the teachers of mathematics to considerable activity. Committees to study the merits of the plan and to make recommendations were appointed by the Central Association of Science and Mathematics Teachers. The reports were received favorably by the teachers. Writers of textbooks undertook the task of putting the new ideas into practice. Their efforts have resulted in suggestions for a variety of new courses designated as combined, composite, correlated, unified, general, mixed, fused, and co-operative mathematics.

To be sure there are distinctions among these courses, but all have the common purpose of breaking down artificial barriers existing between the mathematical subjects. At the present, the plan of unifying and correlating is still the most promising reform that has been offered to improve the mathematical curriculum. It has the indorsement of the National Committee, which says that the plan of breaking down the barriers separating the mathematical subjects

> enables the pupil to gain a broad view of the whole field of elementary mathematics early in his high school course. In view of the large number of pupils who drop out of school at the end of the eighth or ninth school year or who for other reasons then cease their study of mathematics, this fact offers a weighty advantage over the older types of organization under which pupils study algebra alone during the ninth school year to the complete exclusion of all contact with geometry.[5]

Advantages to be derived from correlation.—Experience has shown that correlation offers certain advantages which cannot be obtained by teaching the subjects separately. In the pages which follow attention will be called to some of the more important values to be derived from the plan.

Correlation facilitates the adaptation of mathematics to the needs of the learner. Early adolescence is a period of exploration and discovery, a time in which the pupil should be helped to find himself and to discover where his interests lie and what his tastes and abilities are. It is also the time when he lays the foundation on which to build his future education. A subject selected for instruction should therefore contribute to the greatest possible extent toward making the pupil's experiences in his everyday life and in

[5] *Report of the Committee on Mathematical Requirements* (1923), pp. 12, 13.

the other school subjects meaningful. It should bring the learner into daily contact with life-experiences which help him to adjust himself to his environment.

Each of the mathematical subjects contributes to the intellectual life of the pupil. The first consideration in the arrangement of materials is, therefore, to be given to the plan which secures for the pupil the greatest variety of experiences and activities applicable to his everyday needs. Moreover, the pupils need all of the mathematical subjects in other school studies. For this reason the subject matter of mathematics offered for instruction should be taken from more than one field of mathematics, i.e., from arithmetic, algebra, geometry, and trigonometry. It is a mistake to keep the pupil waiting. As soon as it is feasible he should be taught enough algebra to prefer the algebraic method of solving problems to the less effective arithmetical method, and he should acquire the fundamental facts of geometry and trigonometry which he is likely to make use of in his other studies.

Correlation facilitates the psychological arrangement of subject matter and the adaptation of mathematics to the mental development of the learner. The importance of developing courses in mathematics which conform to the mental development of the pupil cannot be overestimated. The success of any course depends largely on the extent to which this adaptation is made. Algebra, geometry, and trigonometry all contain simple and complex principles. The first step in the direction of a pyschological arrangement is to organize subject matter according to difficulty. This is easily accomplished by bringing together at the beginning the simple facts and principles of mathematics and by increasing gradually the complexity as the pupil progresses. The study of the simple principles of all the subjects is more suitable for the beginning than an intensive study of one subject to the exclusion of all others. The question is not whether one subject is harder than another, but how to arrange in order of difficulty the materials of both. For example, in geometry one of the simplest concepts is the straight line. It may therefore be studied first. The literal number, being the simplest concept in algebra, may be introduced in connection with the idea of unknown length. When the meaning of literal number is understood, combinations such as $3a$ and $a+b+c$ may be intro-

duced as denoting sums of line segments and perimeters of polygons. Simple equations, such as $6x=12$, follow in connection with perimeter problems of regular polygons. For a long period all the algebraic work centers entirely around linear terms, binomials, and polynomials, while the geometric work takes up the angle, triangle, and quadrilateral. When linear expressions have been made the subject of extended and thorough study, quadratic expressions may be taught in an easy and careful manner by relating them to the study of areas of geometric figures. They in turn are followed by cubic expressions denoting volumes, and finally the higher powers are taught. Because of the close correspondence between length, area, and volume in geometry and linear, quadratic, and cubic functions in algebra, the two subjects can be taught and learned together in an easy and natural way.

Furthermore, when the pupil studies simultaneously more than one mathematical subject, progress will be less rapid than in the traditional course where, within the first few weeks, the pupil encounters literal numbers, signed numbers, equations, powers, and even irrational expressions. The content of the first-year algebra course will be distributed over a longer period and the mode of instruction will be less formal.

In traditional algebra and geometry certain concepts and processes, each of which represents a real difficulty to the beginner, are taught together because they belong together logically. The pupil should meet such difficulties gradually and, if possible, one at a time. This is easily arranged when the mathematical subjects are correlated. For example, in connection with the formulas for finding perimeters, areas, and volumes geometry may be used to develop considerable ability in the operations with unsigned literal numbers long before the laws of signs and the operations with positive and negative numbers are studied.

Correlation simplifies the problem of making mathematics concrete. In the early stage of the study of mathematics the concrete should precede the abstract. Content must come before the symbol, thought before the word, and experience before the definition if concepts and processes are to be endowed with meaning. Algebra and geometry as they are commonly taught are made difficult when abstractions are introduced too soon and too suddenly. This might

have been justifiable when both subjects were taught in college largely because of the training the student was to obtain. However, when the subjects are taught in the early grades of the secondary school, logical training remains no longer of principal importance. In secondary mathematics more attention has to be given to the ways by which pupils learn best. In adjusting the mathematical courses to young pupils abstract subject matter should be supplemented by, and connected with, matters of thoroughly concrete character to secure thorough understanding.

When the mathematical subjects are unified, the demand for concrete mathematics is easily satisfied. Geometry furnishes the most simple and most available material with which to make the abstract definitions, rules, and processes of algebra concrete. To the beginner even the simplest elementary concepts of algebra are abstract. Much of the commonly observed confusion about the meaning of such elementary concepts as $2n$, n^2, $(a+b)^2$, and others is caused largely by lack of sufficient concrete experience with them. Because of this same omission pupils have undue difficulty with transposition of terms, the laws of signs, and other fundamentals which, if not understood, make algebra little more than juggling symbols without insight into the real meaning of what is being done. If concrete geometry is combined with abstract algebra, it is invariably found that better comprehension, greater interest, and more lasting results are attained.

Correlation aids understanding. The history of algebra shows that signed numbers were exceedingly difficult to understand until they were made concrete by means of the number scale. The operations with signed numbers introduce a difficulty not experienced with literal numbers alone. The geometric method rationalizes the processes with signed numbers. It explains the steps in the process of extracting the square root of a number. In factoring it throws light upon the way algebraic numbers are made up. On the other hand, algebraic notation and processes are helpful in the study and assimilation of the instructional geometric materials. Arithmetic, algebra, and geometry supplement and reinforce one another. All three are used to express facts about quantity. The formula, the table, and the graph are only different ways of expressing numerical facts. When the three forms of thought are correlated in a

single course of instruction, the student's comprehension of quantity is deepened and simplified. With clear understanding comes increased interest just as surely as the pupil loses interest if he cannot understand a subject, no matter how useful it may be. The objections of pupils to mathematics cease when they understand the work.

Correlation increases mathematical power. Knowledge gained in an abstract subject is easy to recall when made concrete. Pupils fail in some of the most fundamental facts because they have nothing concrete with which to associate them. A common illustration is the expansion of $(a+b)^2$. The tendency is to leave out the middle term, because the most simple thing to do seems to be to move the exponent over the a and b. The tendency is minimized and often entirely checked by teaching pupils to associate the parts $(a+b)^2$, a^2, b^2, and $2ab$ with a diagram which visualizes them. Likewise, the fundamental identities of trigonometry are always difficult to retain until the pupils have learned to represent them concretely by geometric figures, as shown in Figure 19.

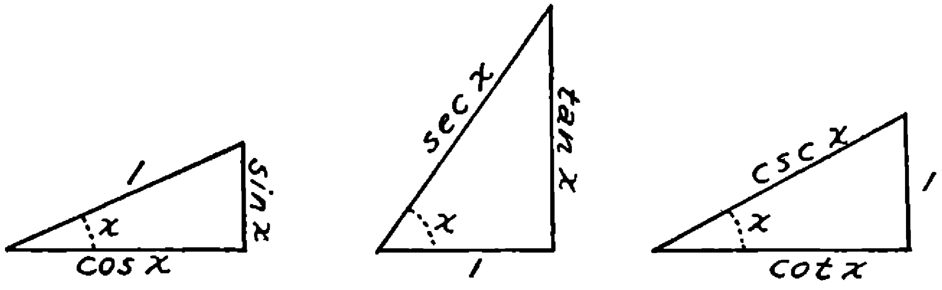


Fig. 19

Showing:

$\sin^2 x+\cos^2 x=1$

$\tan x=\dfrac{\sin x}{\cos x}$

$\sec x=\dfrac{1}{\cos x}$

$\csc x=\dfrac{1}{\sin x}$

Showing:

$\tan^2 x+1=\sec^2 x$

$\cos x=\dfrac{1}{\sec x}$

Showing:

$\cot^2 x+1=\csc^2 x$

$\sin x=\dfrac{1}{\csc x}$

The drawings are easily kept in mind and the equations are readily derived from the drawings.

In the composite type of mathematics and within the various units the learner's experiences are not restricted to one, but are extended freely over all the mathematical subjects, wherever they contribute to the understanding of a particular unit of instruction. Hence, pupils learn in time to attack problems by various methods. For example, a problem in indirect measurement, such as finding the height of a flag pole h, Figure 20, may be solved: (1) geometrically by means of scale drawing; (2) algebraically by solving the equation $h^2+100^2=300^2$; or (3) trigonometrically by means of the formula $h=100 \tan A$. When such problems are encountered in the classroom or elsewhere they are not classified as algebraic, geometric, or arithmetical, but are attacked by whatever method seems the most promising for obtaining the desired solution. Frequently several methods are tried. In fact, the situations in some problems make it necessary to draw for help from several of the mathematical subjects. The pupil who has studied the correlation type of mathematics has therefore an advantage over the one who has studied the subjects separately. When he fails to solve a problem by one method, he will readily try another. An interesting instance of this kind is reported by Stone.[6] A difficult puzzle problem had been submitted for solution to the students of a technical college, and within two weeks four solutions had been turned in, all of which were incorrect. The problem was later submitted to high-school pupils who were studying correlated mathematics, with the result that twenty-four solutions were received, eighteen of which were correct. The eighteen correct solutions were solved by eight different methods, which shows how resourceful pupils may become when trained in courses in unified mathematics.

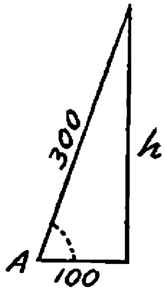


FIG. 20

The habit of using various methods of attack and the experiences of meeting problems in a large variety of situations form an excellent preparation for future college work. The principal reason

[6] Charles A. Stone, "Correlation of the Mathematical Subjects Develops Mathematical Power," *Mathematics Teacher*, XVI (May, 1923), 302–10.

why college teachers find that high-school graduates have not mastered the fundamentals of high-school mathematics are mainly traced to the limited field of application. In algebra a pupil may solve quadratic equations like $2x^2+3x+1=0$ with excellent results, but he may be completely at a loss as to how to solve the equation $2 \sin^2 x+3 \sin x+1=0$ in a course in trigonometry, or the equation $.6t^2-pt+p=0$ in physics. In courses in which algebra, geometry, and trigonometry are correlated, he constantly uses the fundamentals of all high-school mathematics, and the variety of situations is larger than is possible in a single subject. Therefore, he attains better understanding and becomes more resourceful. The situation is analogous to that in arithmetic. Tests given in elementary-school classes show that pupils know arithmetic, but the same pupils later make arithmetical mistakes in the simplest processes and sometimes they are unable even to recall certain arithmetical processes when they occur in algebra. The remedy is to be sought in providing more frequent uses of arithmetic in real and varied applications.

The pupil quickly learns the advantages of knowing and using various modes of treating the facts of quantity. He sees that the graph brings out some facts not easily detected from inspection of a table, while the formula gives all the facts in the briefest form. He sees that the algebraic method is superior to the geometric method, which in turn is superior to the arithmetical method. For example, a table of figures showing the consumption of electricity for each month during the twelve months of a year does not readily reveal certain facts which are easily detected by looking at a graph representing the data of the table. The graph not only gives the data, but yields additional information. It shows that the amount of use of electricity changes, and that there is a gradual increase from month to month which is followed by a decrease. It leads the observer to inquire into the causes for the changes. It may even enable him to tell in advance what the approximate consumption is going to be for the months not designated in the graph.

Similarly, a trigonometric table of sines does not give complete information regarding the changes of the sine function as the angle changes from 0° to 360° as readily as the graph which pictures the changes very clearly.

Moreover, the formula yields all the data given in the table and by the graph, and may be used to give additional data. Thus the graph of the equation $y=3x+5$ furnishes "approximate" values of y, corresponding to values of x, but the equation may be used to find "exact" values of y corresponding to assigned values of x.

Correlation saves time. The importance of saving the time of pupils, particularly of those who wish to continue the study of mathematics, cannot be overemphasized. Today the young men who are to become leaders in the applied sciences need to get to the study of the advanced mathematics as early as possible. Yet they are held back during the secondary-school period. In the past the greatest waste of time has been in the seventh and eighth grades. Here two full years have been devoted almost exclusively to arithmetic and to numerous applications taken from adult life that are of little value because they involve situations unreal to the pupils of these grades and which consequently are not understood. The first year of the senior high school has been devoted entirely to algebra which the pupil promptly forgets while he is studying geometry the next. The practice has been wasteful as to time and energy.

Correlation saves time because the various mathematical subjects are introduced at an early stage and kept in continuous use for a long period of time. Instead of dropping algebra at the end of a year, the pupil continues to employ it and to apply it to geometrical relationships. Algebra simplifies the proofs and exercises of geometry. Skills once developed are kept intact and the frequent and needless reviews which are usually given at the beginning of the third-year course become unnecessary.

Correlation is conducive to functional thinking. The importance of training in functional thinking is being more and more recognized. The idea of relationship should be introduced in the lower grades of the junior high school and the pupil should be taught to think "functionally" very early by keeping the idea predominant in his mind. In arithmetic, such variable numbers as cost, interest, and percentage may be presented as algebraic functions of the number of articles, time, and base. Graphs should be used to picture clearly the relations between the variables expressed by an equation or formula. Literal numbers should no longer be looked upon

merely as fixed unknown numbers, but as variables. The functional character of perimeters, areas, arcs, and angles should be stressed and the functional relations should be expressed by formulas and equations. Many geometric theorems offer opportunity for expressing numerical relations. Formulas for measuring angles in terms of arcs, relations between the side of a regular polygon and the radius of the inscribed or circumscribed circle, formulas for finding areas and perimeters, the theorem of Pythagoras and its generalizations, relations between segments of chords and secants, are all functions to be stated and studied algebraically. In this manner functional correspondence and functional thinking should be constantly emphasized. Indeed, without the study of functions and their geometric representations showing how number and space illustrate each other, how algebra and geometry aid each other, there can be no real appreciation of the meaning of mathematics.

Objections to correlation.—It has been shown that many advantages may be derived from correlation. However, the plan has not been free from criticisms. The following are some of the major objections and answers to criticisms:

1. Many administrators are unwilling to disturb the mathematics curriculum in their schools. They do not disapprove of the plan but prefer to wait until it is more widely accepted. When that time comes they will be willing to make the change. It is evident that progress on any change, no matter how beneficial it might be, would be retarded or even impossible if all administrators would take this position.

2. The transfer of pupils from one school to another is made difficult if the courses in the schools are not alike. This objection is not nearly as serious today as formerly because even the courses labeled "algebra" are no longer identical. The newer courses in algebra and geometry are sufficiently different to make it inadvisable to transfer or accept pupils from other schools without careful examination of their previous training.

3. It is claimed that teachers of mathematics are not well-enough prepared to teach the combination courses. This is a serious reflection upon the training of the teachers. If they are not prepared to teach the combined courses, they are also not prepared to teach the separate courses. No new type of mathematics has been

discovered. The teachers who really know how to teach arithmetic, algebra, geometry, and trigonometry separately will be qualified and able to teach these subjects when they are presented in combination form.

4. Enthusiasts, it is said, have allowed themselves to go too far in the process of unifying and have mixed the mathematical subjects in an unnatural way. This is an argument for better correlation rather than against correlation.

5. Correlation is said to rob algebra and geometry of their individualities. Clear demarcation is not an important matter in the lower levels. Understanding, after all, must be given first consideration. Correlation, however, does not intend to eliminate all distinctions between algebra and geometry to the point where they cannot be told apart.

6. Teachers complain that correlation adds difficulties in teaching. For example, they refer to the graphical solution of simultaneous equations, the geometrical illustrations of the laws of signs, and the multiplication of polynomials by means of rectangles. The difficulty usually arises when teachers make the false assumption that all geometry is concrete. Geometry used to illustrate algebra must be understood to be concrete. The following experience illustrates the point. A teacher demonstrated the teaching of the laws of signs to a class of high-school pupils in the presence of the parents. Line segments were used to make clear the meaning of the laws. After the lesson had been finished and the class dismissed the parents remained for discussion of the demonstration. It became apparent during the discussion that the parents had failed to understand the lesson which had offered no difficulty at all to the pupils. The reason was easily found. The pupils had previously studied line segments and to them the use of lines made the abstract laws of signs concrete. The parents were lacking in familiarity with lines. To them lines were not concrete and therefore of no assistance in understanding the laws of signs. The use of lines actually increased their difficulty.

Tests have shown that the use of rectangles does not help a class to grasp the law of multiplying algebraic polynomials unless the particular properties and principles of the rectangle to be used have been studied and understood. The same principle applies to the

graphical solutions of equations. The use of the method presupposes a knowledge of graphical methods. Only facts that are familiar and understood are concrete. If geometry has not been carefully studied, the use of geometric material in courses of algebra is of questionable value.

Measured results of correlation.—Several investigators have undertaken the task of measuring the success of correlation. In the laboratory schools of the University of Chicago, which have used correlated mathematics for a period of years, the following results have been reported:[7]

1. Correlation has accomplished a considerable saving of time for the pupils. In the three years beginning with the seventh grade they cover a year's work in algebra, intuitive geometry, and a year's work in logical geometry in addition to the arithmetic usually offered in Grades VII and VIII.

2. Classes having studied correlated mathematics in Grades VII and VIII show superiority in arithmetic over classes having devoted the entire time to arithmetic.

3. When tested in the fundamentals of algebra and geometry, ninth-grade pupils who have studied correlated mathematics compare favorably with tenth-grade pupils who studied first-year algebra and plane geometry.

4. The work of University High School graduates taking mathematics at the University of Chicago is uniformly higher than the class averages.

5. University High School graduates taking mathematics at other universities succeed better than the average student.

Bowyer[8] made a study of the results of correlated mathematics measured by tests on algebra and geometry. He found very little difference in the results at the end of the first year. However, at the end of the second and third years the advantage seemed to be decidedly in favor of the correlated mathematics group of pupils. Part of Bowyer's study was concerned with the quality of work done by University High School graduates in courses in college

[7] E. R. Breslich, *A Critical Examination and Experimental Reconstruction of Secondary Mathematics* (Doctor's dissertation, University of Chicago, 1926).

[8] L. V. Bowyer, *A Study of the Results of High School Instruction in Correlated Mathematics* (Master's thesis, University of Chicago, 1923).

mathematics. He found again that the advantages seemed to be on the side of the University High School graduates. He concluded that superior quality of work was either the result of better teaching in the University High School or that pupils having studied correlated mathematics in the high school were better prepared for college mathematics than those who studied the separate courses.

Burks[9] compared a general mathematics group of pupils with an algebra-geometry group. The two groups were comparable in intelligence and scholarship. He used a standardized algebra test, a standardized geometry test, and a special test made by a group of high-school teachers. He found that the general mathematics group ranked higher than the algebra-geometry group in each of the tests. Furthermore, more pupils of the first group continued the study of mathematics in the third year, and the quality of their work was superior to that of the second group. Burks reports for the first group a saving of time and high interest on the part of the pupils.

Reeve[10] reports that graduates of the University of Minnesota High School who had at least two years of general mathematics in the high school were more successful in college mathematics than those who did not have two years of general mathematics.

Crow and Dvořák,[11] after conducting an experiment to compare general mathematics with the algebra-geometry plan, came to the conclusion that pupils who have studied reorganized mathematics when measured by tests in algebra and geometry disclose as much knowledge in algebra and geometry as pupils who were taught according to the old type of organization. They express the view that with well-trained teachers and improved textbooks better results would be attained.

The foregoing studies seem to agree that when measured by tests designed for pupils who have studied algebra and geometry pupils who have studied correlated mathematics attain results at least as good as those who are taught in separate courses. Pupils in the

[9] W. D. Burks, "An Experiment Comparing the Efficiency of General Mathematics with Algebra and Geometry," *Mathematics Teacher*, XVII (October, 1924), 343–47.

[10] W. D. Reeve, "The Case for General Mathematics," *ibid.*, XV (November, 1922), 381–91.

[11] Jane M. Crow and August Dvořák, "A Study of Achievement in General Mathematics," *School Science and Mathematics*, XXIX (January, 1929), 21–26.

University of Chicago High School are being regularly tested, at least once each semester, by means of algebra and geometry tests. Invariably it is found that they rank somewhat above the medians of these tests. No violence seems to be done to their algebraic and geometric training. However, they have enjoyed all the advantages which were described in the foregoing pages and which may be derived from the correlation plan.

Some of the foregoing investigations show that pupils who study correlated mathematics gain from year to year in algebraic and geometric achievement over those who study the courses separately. This factor is significant in view of the fact that practically all investigations which have measured the amount of algebra and geometry actually retained by pupils who have studied the subjects separately have found the amount surprisingly small. In fact, so little seems to be permanently retained that it is hard to see how anything is to be risked by a change to almost any other than the algebra-geometry plan.

It is easily seen that tests in traditional algebra and geometry give an advantage to pupils who have been trained in the separate courses. As a rule, courses in combined mathematics minimize the purely mechanical manipulative phases of algebra which these tests stress. They devote more time to the development of understanding and mathematical power. Whenever pupils are measured by tests of mathematical power, it seems that groups trained in the correlated type of mathematics show decided resourcefulness and superior training. An illustration of this fact has been described by Stone.[12] He summarizes his findings by concluding that pupils trained under a plan in which the mathematical subjects are correlated possess power of resourcefulness, that subject matter and methods become valuable tools in new situations, and that further study of mathematics is being motivated.

In a recent study which aimed to measure the development of functional thinking,[13] comparisons were made between two groups of pupils, one of which had been taught by the traditional algebra-

[12] Stone, *loc. cit.*

[13] E. R. Breslich, "Measuring the Development of Functional Thinking in Algebra," *The Teaching of Algebra* (Seventh Yearbook, National Council of Teachers of Mathematics, 1932).

geometry plan while the other had studied correlated mathematics. In the algebra and geometry tests the medians of the classes of the second group were all somewhat above the median of the first group. However, there was marked superiority of the second group over the first in the results with the functional thinking test, as shown in Table XXXIV.

It seems that growth in functional thinking is comparatively rapid among pupils who study correlated mathematics. Further investigation may show that certain of the primary values of

TABLE XXXIV

Grades	Medians of Algebra-Geometry Group Representing Nine Schools	Medians of the Correlated-Mathematics Group
IX	22	23
X	23	35
XI	31	67
XII	36	78

mathematics are being attained by the correlation plan to an extent not possible with the separate courses.

Correlation of algebra and geometry with arithmetic.—An analysis of the content of junior high school algebra and geometry discloses numerous opportunities for teaching the arithmetical processes and for practice. Almost every topic employs one or several operations with whole numbers or with fractions, both common and decimal. The results of the analysis have been tabulated in Table XXXV. The table makes it apparent that pupils who are deficient in arithmetic will often find their progress in secondary-school mathematics seriously retarded. To the teacher the table indicates definite places in the course where the various processes may be retaught. It presents a plan for incidental teaching of arithmetic and shows where practice in arithmetical manipulation may be provided.

Organizing correlated mathematics for teaching.—Previous chapters have shown how the instructional material of algebra and geometry may be organized for purposes of instruction. Several examples will now be presented to illustrate an organization in which the materials of arithmetic, algebra, and geometry are correlated. Advantages will be taken of the natural relationships exist-

TABLE XXXV

A Plan Providing for Training in Arithmetic in Secondary-School Mathematics

Algebraic and Geometric Activities	Integers	Fractions			Square Root	Units of Measure	Abbreviated Processes; Approximating Values	Per Cent
		Common	Mixed	Decimal				
Measuring line segments and drawing lines of given lengths	Reading large numbers; rounding off large numbers	Fractions approximated to the nearest sixteenth of an inch	Eighths of an inch; sixteenths of an inch	Meaning of decimal fraction; reading; approximating to the nearest tenth of a centimeter	....	Becoming familiar with the metric and English systems	....	Measuring by the hundred; meaning of per cent; using lines to picture per cent
Finding average lengths and other averages	Adding; dividing	Adding; dividing	Adding and dividing mixed numbers	Adding; dividing	....	A large variety of units is used	Stating results to the nearest tenth or hundredth	Average per cent
Finding ratios of line segments	Dividing	Dividing; reducing	Dividing; reducing	Dividing	....	A large variety of units is used	Dividing to the nearest tenth or hundredth	Per cent one line is of another
Interpreting and making statistical graphs	Reading and rounding off large numbers	Changing common fractions to decimal fractions	Changing mixed numbers to decimal fractions	Reading; rounding off to nearest tenths	....	Using the metric system, the English system; a large variety of units is employed	....	Reading per cents
Making graphs of precise mathematical laws	Multiplying; dividing	Reducing common fractions to decimals	Reducing to decimal fractions	Multiplying; dividing	....	....	Multiplying and dividing to the nearest tenth	....
Making and reading circular graphs	Reading and rounding off large numbers	Multiplying by a whole number; reducing fractions	....	Multiplying; dividing; adding	....	Degree, minute	....	Finding per cents
Making scale drawings	Multiplying; dividing	Multiplying by a whole number; dividing by a whole number	Multiplying and dividing by scale	Multiplying by scale; dividing by scale	....	Units of length; degree, minute	....	....
Applying the principles of similarity and proportion	Finding ratios; multiplying; dividing	Multiplying fractions by whole numbers; reducing	Reducing	Multiplying; dividing	....	Units of length	Finding ratios to the nearest tenth or hundredth	....

TABLE XXXV—*Continued*

Algebraic and Geometric Activities	Integers	Fractions			Square Root	Units of Measure	Abbreviated Processes; Approximating Values	Per Cent
		Common	Mixed	Decimal				
Finding and using trigonometric ratios	Dividing; multiplying	Reducing; multiplying fractions by whole numbers	Dividing	Multiplying; dividing	....	Units of length	Changing common fractions to decimal fractions	....
Solving circumference problems	Multiplying; dividing	Multiplying; dividing	Reducing; multiplying	Multiplying; dividing	....	Units of length	Multiplying and dividing a given number of figures	....
Measuring surfaces	Multiplying; dividing	Multiplying; dividing	Changing to fractions, and fractions to mixed numbers	Multiplying; dividing	Finding square root by factoring; extracting square root; table of square roots	Square inch; square centimeter; square yard; square mile	Multiplying; dividing	Meaning of per cent; representing per cents by surfaces
Using the theorem of Pythagoras in problems	Adding; subtracting; multiplying	Multiplying; adding; subtracting	Multiplying; adding; subtracting	Multiplying; adding; subtracting	Finding square roots by factoring and by table	Units of surface	Multiplying; dividing	....
Finding areas by formulas	All fundamental processes	All fundamental processes with fractions are used	Changing to common and decimal fractions	All fundamental processes	Extracting square root; tables of roots	Units of surface	Multiplying; dividing	....
Finding volumes by formulas	All fundamental processes	All fundamental processes	Changing to common and decimal fractions	All fundamental processes	Extracting square root; tables of roots	Units of volume	Multiplying and dividing	....
Solving equations of the form $ax=b$	Multiplying; dividing	Multiplying; dividing	Dividing by mixed numbers	Multiplying; dividing	....	....	Dividing	....
Solving equations of the form $ax+b=c$	Adding; subtracting; multiplying; dividing	Adding; subtracting; multiplying; dividing	Changing to common and decimal fractions	Adding; subtracting; multiplying; dividing	....	....	Dividing	....
Solving quadratic equations	All fundamental processes	All fundamental processes	....	All fundamental processes	Square roots found by extracting, by table, and by factoring	....	Multiplying and dividing	....

TABLE XXXV—*Continued*

Algebraic and Geometric Activities	Integers	Fractions			Square Root	Units of Measure	Abbreviated Processes; Approximating Values	Per Cent
		Common	Mixed	Decimal				
Solving problems by means of formulas of percentage, interest, and discount	Multiplying; dividing	Multiplying; dividing	Changing to common and decimal fractions; multiplying	Multiplying; dividing	....	....	Multiplying and dividing	Percentage problems
Combining similar terms	Adding and subtracting	Adding and subtracting	Adding and subtracting	Adding and subtracting	....	....	....	....
Evaluating polynomials and formulas	All fundamental processes	All fundamental processes	All fundamental processes	All fundamental processes	....	....	....	....
Checking equations	All fundamental processes	All fundamental processes	All fundamental processes	All fundamental processes	....	....	....	....
Performing algebraic processes	All fundamental processes	All fundamental processes	All fundamental processes	All fundamental processes	....	....	....	....
Checking algebraic processes	All fundamental processes	All fundamental processes	All fundamental processes	All fundamental processes	....	....	....	....

ing among the mathematical subjects. Thus, arithmetic, algebra, and geometry are used to represent quantitative facts and relations in different ways. The first employs arithmetic figures and tables, the second uses letters and formulas, and the third pictures quantitative facts by means of diagrams and graphs. In the early stages, which present the mathematics to be required of all pupils, geometry may supply the basic materials for instruction, algebra and arithmetic functioning as aids in the study and manipulation of the geometric materials.

The assimilative materials of the first unit must necessarily be of a simple type. Hence, in the beginning lines are used, then polygons formed by line segments, and finally angles. Information is gained from the activities of measuring and drawing. In most cases they are to be performed by the pupils themselves. Data thus obtained mean more to the learner than data derived from measurements performed by someone else.

The use of ruler, compasses, and protractor develops skill in manipulating these instruments. The employment of a variety of units of measure familiarizes the pupils with the meanings and sizes of such units. He attains a feeling and appreciation of the degree of accuracy to be expected in measurement.

The arithmetical processes are being kept in constant use. Meaning is attached to each new algebraic symbol as it is introduced. Proficiency in the algebraic processes is developed through actual use. Gradually the pupil acquires a vocabulary of the frequently used algebraic and geometric terms.

A detailed plan of the organization of the first unit is found in Table XXXVI.

The table shows the gradual development of algebra as it is needed in the geometric work of the unit. Literal numbers, formulas, and equations are closely correlated with the study of lines and angles. The equations are all of a simple type, the most difficult being of the form $ax+bx=c$, where a, b, c are whole numbers or fractions. The technique of solving verbal problems is being slowly developed.

The activities of measuring and drawing line segments are of much importance. They are basic not only to geometry, but should be looked upon as the beginning of graphical work and of analytic

TABLE XXXVI*

DEVELOPING AN UNDERSTANDING OF LINES, ANGLES, AND LINEAR POLYNOMIALS

Pupil Activities	Geometric Concepts	Skill with Instruments	Geometric Principles	Algebraic Concepts	Formulas; Equations	Algebraic Processes	Algebraic Principles	Relationships	Graphs	Units	Arithmetical Concepts	Arithmetical Processes	Uses	Historical Discussion
Measuring line segments	Geometric line; approximate length; line segment; number representation by segments	Inch ruler; metric ruler; compasses; squared paper	......	Unknown number and literal number denoting length	....	....	......	....	....	Inch; centimeter	Common and decimal fractions; mixed numbers; averages; approximate value	Addition, subtraction of mixed numbers, and fractions	Distances on maps; measuring distances in home, park, and school yard	History of the metric system
Measuring line segments in drawings, designs, and graphs	Scales in designs, drawings, and maps	Ruler; compasses	......	....	....	....	......	....	Interpreting graphs	Per cent; inch; foot	Representation of numbers in tables	Multiplication of mixed numbers and fractions	Per cents in daily life; measuring lines in designs, blueprints, floor plans, and maps	....
Comparing measured line segments	Ratio of line segments	....	......	Ratio of unknown numbers, as $\frac{a}{b}$	....	Reducing ratios and fractions	......	Comparing lengths; times as great	Interpreting graphs	......	Ratio of two numbers	Division of decimal fractions; approximate division; reduction of common fractions	....	Development of the literal number
Drawing line segments of given lengths	Representation of a number by a segment	Ruler; compasses; squared paper	A†; B‡	....	....	....	......	Interpreting relations in graphs	Making and interpreting bar graphs and line graphs	Inch; foot; yard; mile; dollar; pound; hours; degrees	Tabular representation of numbers; large numbers; decimal and common fractions; approximate values	Changing units; reading large numbers; rounding off large numbers	In interpreting tables; making scale drawings, maps, and floor plans; graphs in science and in shop work	....

* For a detailed discussion see *Seventh-Year Mathematics*, chaps. i and iii.

† A. Two straight lines can have only one point in common.

‡ B. Through two points only one straight line can be drawn.

TABLE XXXVI—*Continued*

Pupil Activities	Geometric Concepts	Skill with Instruments	Geometric Principles	Algebraic Concepts	Formulas; Equations	Algebraic Processes	Algebraic Principles	Relationships	Graphs	Units	Arithmetical Concepts	Arithmetical Processes	Uses	Historical Discussion
Drawing polygons	Triangle; quadrilateral; pentagon; hexagon; equilateral polygon	Straight edge; compasses	......	....	....	....	......	....	....	......	....	....	....	....
Finding perimeters	Perimeter	Straight edge; compasses; squared paper	......	Monomial; sum, as $a+b+c$; polynomial; product, as ab; formula; equation	Perimeter formula, as $p=6x$; distance formula, as $d=30t$	Making formulas; translating formulas; evaluating formulas; adding and subtracting terms	C§	In formulas; relations in table and in graph	....	......	....	Evaluation of polynomials; all fundamental processes	....	....
Solving perimeter equations	....	....	......	Polynomial, as $a+b+c+d$; trinomial; parentheses; similar terms; coefficient; equation	$120=6x$	Solving $120=6x$; solving $8x-4x=30$	D‖	In equations; in formulas	....	......	Approximate values	Reducing fractions; adding and subtracting whole numbers and fractions; dividing	Solving problems by means of equations	Development of the equation

§ C. The law of order in addition and subtraction.
‖ D. The division axiom.

TABLE XXXVI—*Continued*

Pupil Activities	Geometric Concepts	Skill with Instruments	Symbols and Notation	Geometric Principles	Algebraic Concepts	Formulas; Equations	Algebraic Processes	Relationships	Units	Arithmetical Processes	Uses	Historical Discussion
Observing angles	Angle; side; arm; vertex	…	…	…	…	…	…	…	…	…	In the classroom; in geometric drawings; in surveying, navigation, astronomy	…
Denoting angles	Right, straight, acute, obtuse angle	…	$<ABC$, $<B$, or b; R.A. for "right angle"	…	Unknown number; variable; general number	…	…	…	…	…	In diagrams	…
Measuring angles	Size of angle	Protractor; straight edge	…	…	…	…	…	Changing size of angles	Degrees; minutes; seconds	…	In scale drawings; out of doors; in surveying; in geography	History of angle units and angular measurement
Studying angle relations	Adjacent angles	…	Literal number, as a	…	Equation; unknown; number	…	…	…	…	…	…	…
	Perpendicular lines	Protractor; compasses	$\perp$ for "is perpendicular to"	E¶	Relation aspect of equation	$m=n$ for perpendicularity	…	Adjacent angles: $m=n$	…	…	…	…
	Supplementary angles	…	$a+b=180$ for supplementary	…	…	Solving $4x+x=180$	Combining terms	$a+b=180$ for supplementary angles	…	Adding; subtracting; dividing	…	…
	Complementary angles	…	$a+b=90$ for complementary	…	…	Solving $8x+x=90$; solving verbal problems	Evaluating in checking	$a+b=90$ for complementary angles	…	…	…	…
	Opposite angles	…	$a=b$	F**	…	…	…	$a=b$ for opposite angles	…	…	…	…

¶ E. If two lines are perpendicular, the adjacent angles are equal.
**F. If two lines intersect, the opposite angles are equal.

TABLE XXXVI—*Continued*

Pupil Activities	Geometric Concepts	Skill with Instruments	Symbols and Notation	Geometric Principles	Algebraic Concepts	Formulas; Equations	Algebraic Processes	Relationships	Units	Arithmetical Processes	Uses	Historical Discussion
Estimating size of angle	Approximate measure	....	....	......	....	....	....	....	....	....	....	....
Drawing angles of given size	....	Protractor; compasses	....	......	....	....	....	....	....	....	In shop work	....
Drawing parallel lines	Parallel lines; corresponding angles; alternate interior angles; interior angles on the same side	Protractor; squared paper; compasses	Use of ‖ for "is parallel to"; literal numbers	G††	....	Solving verbal problems	....	Corresponding angles; interior angles; alternate interior angles	....	Adding; subtracting; dividing	In designs; in making architect plans; in mechanical drawing	....
Establishing facts by informal reasoning	....	....	....	G††; H‡‡	....	....	Substituting; subtracting equals from equals	....	....	....	....	....

†† G. If two parallel lines are cut by a third line, the corresponding angles are equal; the alternate interior are equal; the interior angles on the same side are supplementary.

‡‡ H. Things equal to the same thing are equal to each other.

geometry. The finding of ratios of line segments forms an easy and natural introduction to similarity and trigonometry.

Arithmetic is kept in almost constant use, as it occurs in practically all of the activities, and is closely correlated with geometry and algebra.

Training in functional thinking is related to changes of line segments and of angles. Important angle relations are discovered by the pupils.

Table XXXVII shows how algebra, geometry, and arithmetic may be correlated in a unit developing methods of indirect measurement. The central problem, learning how to measure inaccessible distances and angles, supplies the motive for the study. The pupil's interest is aroused because he is becoming acquainted with new methods superior to that of direct measurement. He appreciates the fact that he is gaining power. The unit introduces him in a simple manner to two or the most important concepts of geometry, congruence and similarity. He discovers several new properties of the isosceles, equilateral, and right triangle, and applies them immediately in practical problems.

His knowledge of algebraic equations is extended to include proportions. The multiplication axiom is used for the first time in this unit. Opportunities for functional thinking are provided in the study of the relationships between the angles of right and oblique triangles and in the proportionality of the corresponding sides of similar triangles.

The unit contributes to the further development of the technique of solving verbal problems. Some numerical trigonometry is introduced in a natural way through the use of the tangent ratio in the problem of indirect measurement.

There is ample opportunity for practice in arithmetical computation. Practically every problem in the unit involves the fundamental operations with whole numbers and fractions, especially decimal fractions. The question of figure accuracy and approximation of values is receiving attention.

Table XXXVIII shows how the mathematical subjects may be correlated in a unit whose central theme is the circle. This unit is suitable for pupils finishing the seventh grade or just beginning the eighth. Important properties of the circle are derived from the

TABLE XXXVII*

Developing Methods of Indirect Measurement

Pupil Activities	Geometric Concepts	Skill with Instruments	Symbols and Notation	Geometric Principles	Algebraic Concepts	Formulas; Equations; Problems	Algebraic Processes	Algebraic Principles	Relationships	Arithmetical Concepts	Arithmetical Processes	Uses	Historical Discussion
Drawing triangles	Triangle	Ruler; protractor	△ for "triangle"	....	....	....	....	....	....	....	....	In designs; in architecture	....
Measuring sides and angles of triangles	Isosceles triangle; equilateral triangle; right triangle	Ruler; protractor	Letters for unknown sides and angles; $a=b$ for isosceles; $a=b=c$ for equilateral; $a+b=90$ for complementary angles	A†; B‡; C§; D‖; E¶	Literal number; binomial; trinomial	$a+b+c=180$; $x+3x+6x=180$; $a+b=90$; solving verbal problems	Solving equations like $x+3x+6x=180$; solving $6x=180$; solving $6x+2x=90$	Combining similar terms	Relation between the angles of a triangle; relation between the sides of a right triangle; relation between acute angles of a right triangle	Sum; decimal fractions	Adding, subtracting, and dividing whole numbers and decimal fractions	In surveying; in making scale drawings	....
Making congruent triangles	Congruence	Protractor; compasses; ruler	≅ for "is congruent to"	F**; G††	....	....	....	....	Dependence of size and shape of triangle on given parts	....	....	In solving problems; in indirect measurement; in surveying	History of land surveying

* A detailed discussion is given in *Eighth-Year Mathematics*, chap. vii.

† A. The sum of the angles of a triangle is 180°.

‡ B. If two angles of a triangle are equal, the triangle is isosceles.

§ C. If three angles of a triangle are equal, the triangle is equilateral.

‖ D. The acute angles of a right triangle are complementary.

¶ E. In the 60°–30° right triangle the hypotenuse is twice as long as the shortest side.

** F. Two triangles are congruent if two angles and a side of one are equal to the corresponding parts of the other.

†† G. Two triangles are congruent if two sides and the included angle of one are equal to two sides and the included angle of the other.

TABLE XXXVII—*Continued*

Pupil Activities	Geometric Concepts	Skill with Instruments	Symbols and Notation	Geometric Principles	Algebraic Concepts	Formulas; Equations; Problems	Algebraic Processes	Algebraic Principles	Relationships	Arithmetical Concepts	Arithmetical Processes	Uses	Historical Discussion
Finding distances by scale drawings	Scale drawing; angle of elevation, of depression	Ruler; compasses	....	....	....	....	....	....	Relation between actual size and scale drawing	....	....	Indirect measurement in surveying, in designs, in shop work, in school playground	....
Making similar triangles	Similarity; proportion of line segments	Ruler; compasses; squared paper	∽ for "is similar to"	H‡‡; I§§	Proportion	$\frac{x}{5}=\frac{3}{8}$	Solving $\frac{x}{5}=\frac{3}{8}$	J‖‖; K¶¶	Relations between the parts of similar figures	Ratio; proportion	Division of decimal fractions; multiplication of fractions	Indirect measurement in solving problems	History of proportion
Finding distances with the tangent table	Tangent ratio	Using table	tan x for tangent of x	....	....	$\frac{h}{181}=\tan 65°$	Solving $\frac{h}{181}=\tan 65°$	....	Dependence of angles on the sides of a triangle	Ratio; approximate measure	Multiplication and division of decimal fractions	In indirect measurement; in surveying	History of trigonometric ratios

‡‡ H. Two triangles are similar if the corresponding angles are equal.

§§ I. Two triangles are similar if the corresponding sides are in proportion.

‖‖ J. Multiplication axiom.

¶¶ K. In a proportion the product of the means is equal to the product of the extremes.

TABLE XXXVIII*

The Meaning and Uses of the Circle

Activities	Geometric Concepts	Skill with Instruments	Symbols and Notation	Geometric Principles	Algebraic Concepts	Formulas; Equations	Algebraic Processes	Relationships	Graphs	Units	Arithmetical Concepts	Arithmetical Processes	Uses	Historical Discussion
Drawing circles	Circle; center; radius; diameter; arc	Compasses; ruler	⊙ for circle; Ⓢ for circles; ⌒ for arc	....	Literal numbers denoting lengths of arcs	....	....	Size of circle depends on radius; diameter is twice radius	....	Inch; foot; meter; centimeter	....	....	In designs, machines, architecture; in making isosceles and equilateral triangles	....
Measuring angles by means of arcs	Equal circles; central angle; arc degree	Compasses; protractor	....	A†	....	....	....	Between central angles and arcs	....	....	....	....	....	....
Locating places	Meridians; parallel circles; equator	Compasses	"lat." for latitude; "long." for longitude	....	....	....	....	....	....	....	....	....	....	....
Determining local time	....	....	....	....	....	....	....	Between time and longitude	....	Degree of latitude, of longitude; minute; second	Tabular representation	Subtraction; multiplication of whole numbers and fractions	In geography	....
Reading meters; computing bills	....	Compasses in drawing dials	....	....	....	....	....	Between amount of bill and consumption of gas	....	Dollars; cents; cubic foot	....	All processes with decimal fractions	....	....

* For a detailed organization see *Seventh-Year Mathematics*, chap. iv, and *Eighth-Year Mathematics*, chap. i.

† A. In equal circles equal central angles cut off equal arcs.

TABLE XXXVIII—*Continued*

Activities	Geometric Concepts	Skill with Instruments	Symbols and Notation	Geometric Principles	Algebraic Concepts	Formulas; Equations	Algebraic Processes	Relationships	Graphs	Units	Arithmetical Concepts	Arithmetical Processes	Uses	Historical Discussion
Making and interpreting circular graphs	Circular graph	Protractor; compasses	....	B‡	....	$\frac{n}{360}=\frac{19}{81}$	Solving $\frac{n}{360}=\frac{19}{81}$	Between central angles and numerical facts given in tables	Circular graphs	Degrees	Tabular representation	All processes with decimal fractions	In statistics	....
Making geometric constructions with ruler and compasses	Congruence of triangles; bisection of angle; regular hexagon; perpendiculars; square; altitude; median; concurrent lines	Compasses and straight edge are used to make all fundamental constructions; protractor is used as a checking instrument	$\cong$ for "is congruent to"	C§	....	....	....	....	....	....	....	....	In designs; in geometric diagrams	....
Measuring the length of a circle	Circumference	Squared paper in graph of $c=2\pi r$	Literal number to denote length	D‖	Variable; constant	$c=2\pi r$ $c=\pi d$	Solving verbal problems leading to $c=2\pi r$; solving the equation $c=2\pi r$	Between radius and circumference; direct variation	Graph of $c=2\pi r$	....	Approximate value of π	All processes with common and decimal fractions; approximate products and quotients	In a large variety of problems	Determination of the value of π

‡ B. In the same circle central angles are measured by the intercepted arcs.

§ C. Two triangles are congruent if the sides of one are equal to the sides of the other.

‖ D. The circumference of a circle is equal to 2π times the radius.

activities of drawing and measuring circular lines. Knowledge thus gained is used in making simple designs, geometric constructions with ruler and compasses, and circular graphs. The usefulness of the circle is further illustrated in meter reading and in problems relating to latitude, longitude, and time. The study of congruence of triangles is extended to include the case when the sides of one triangle are equal to the sides of another.

The development and uses of the circumference formula involve many experiences and problems which aid the pupil in understanding the meaning of formulas and equations. Proficiency in problem-solving is increased. The concepts of variable, constant, and functional relationships are introduced in connection with the development of the circumference formula. Simple proportions arise in the problem of making circular graphs.

Much practice in computation with whole numbers and fractions is offered in making graphs representing the formula $c=2\pi r$ and in solving problems in which the formula is used.

It will be noticed that the content of the three units outlined in Tables XXXVI, XXXVII, and XXXVIII is taken from the first level of algebra and geometry. The pupil has learned the meaning of literal number, formula, and linear equation. He has had considerable practice in the technique of problem-solving, graphical representation, and functional thinking. However, the algebraic materials were usually introduced as an aid to the pupil in the study of geometry. He is now prepared to spend some time and effort in systematizing his knowledge of algebra. He will be able to appreciate a unit on formulas and equations of the first degree.

Such a unit should review and extend the study of equations, until the pupil is able to solve any equation which may be classified as belonging to the general type $ax+b=cx+d$, where a, b, c, and d are whole numbers or fractions. New formulas should be added to those taught in the other units, e.g., the percentage and interest formulas. The technique of problem-solving should receive much attention by the introduction of a variety of problems relating to interest, commission, discount, and other applications taken from adult life but falling within the experiences of pupils.

Table XXXIX shows an organization of materials belonging to the second level of algebra and geometry. The measurement of surfaces introduces algebraic expressions of the second degree.

TABLE XXXIX*

Measuring Surfaces of Plane Figures

Pupil Activities	Geometric Concepts	Geometric Principles	Algebraic Concepts	Formulas; Equations; Problems	Algebraic Processes	Algebraic Principles	Relationships	Graphs	Arithmetical Processes	Uses	Historical Discussion
Measuring the sides and angles of a rectangle	Rectangle; base; altitude	A†; B‡	Literal number	....	....	....	....	....	....	....	....
Measuring the surface of a rectangle	Unit of surface; area	C§	Product of two literal numbers	$A=bh$	Evaluation of $ab+cd$, and $ab+cd+ef$	D‖	Area varies as the base, or as the altitude	Making the graph of $A=5h$	Multiplication of whole numbers and fractions; reduction of fractions	Problems relating to daily occupations of adults and to pupil activities	....
Multiplying monomials by polynomials	Product ab represented by area of a rectangle	....	Monomials and polynomials of second degree, as $ab+cd+ef$	Solving equations of the type $10(x+2)=119$; solving verbal problems	The process of multiplying monomial by polynomial is derived by the use of the rectangle	E¶	Relation between factors and products	....	Multiplication by zero; all processes are used in evaluation and checking results	....	....
Multiplying polynomials by polynomials	....	....	Product of two polynomials	....	The process of multiplying polynomials is derived by use of the rectangle	F**	....	....	All processes used in evaluation and checking	....	....

* A detailed discussion is found in *Seventh-Year Mathematics*, chap. vii, and in *Eighth-Year Mathematics*, chaps. ii and iii.

† A. The opposite sides of a rectangle are equal.

‡ B. The diagonals of a rectangle are equal.

§ C. The area of a rectangle is equal to the product of the base by the altitude.

‖ D. The law of order in multiplication.

¶ E. The distributive law of multiplication.

** F. The law for multiplying two polynomials.

TABLE XXXIX—*Continued*

Pupil Activities	Geometric Concepts	Geometric Principles	Algebraic Concepts	Formulas; Equations; Problems	Algebraic Processes	Algebraic Principles	Relationships	Graphs	Arithmetical Processes	Uses	Historical Discussion
Measuring the square; surface of a square	....	G††	$a \cdot a = a^2$	$A = a^2$	....	....	Relation between side and area of square	Making the graph of $A = a^2$	Squaring two-figure numbers, as $(24)^2$	Use of the square in picturing percentage	History of exponents
Squaring a binomial	....	....	Trinomial square; identity	....	Multiplying $(a+b)(a+b)$ by means of a square, by the formula	Law of expanding; $(a+b)^2 = a^2+2ab+b^2$	....	....	....	Use of square in picturing $(a+b)^2$	....
Finding the square root by using the square	....	H‡‡	Radical sign; square root	$a^2 = 100$; $x^2+b^2=c^2$; solving verbal problems	Solving simple quadratic equations like $x^2+36=100$	....	Relation between the sides of a right triangle	....	Extracting square roots	In practical problems	Life and theorem of Pythagoras
Measuring the surfaces of parallelogram, trapezoid, triangle, and circle	Parallelogram; trapezoid; triangle; circle	I§§; J‖‖; K¶¶; L***	....	$A = bh$; $A = \frac{1}{2}h(a+b)$; $A = \frac{1}{2}bh$; $A = \pi r^2$; solving equations like $\frac{7(x+6)}{2} = 63$ and $r^2 = 100$	....	....	Relations contained in formulas for finding areas	....	All processes in manipulation of formulas; abbreviated multiplication and division	An abundance of practical problems	History of the life of Archimedes

†† G. The area of a square is the square of one side.

‡‡ H. The theorem of Pythagoras.

§§ I. The area of a parallelogram is equal to product of base by altitude.

‖‖ J. The area of a trapezoid is equal to product of one-half the altitude by the sum of the bases.

¶¶ K. The area of a triangle is equal to one-half the base by the altitude.

*** L. The area of a circle is equal to π times the square of the radius.

The formulas for finding the areas of plane figures offer numerous opportunities for evaluation, solution of equations, problem-solving, and functional thinking. The relationships involved in the formulas are given attention and several are represented graphically. Equations of the second degree of the following forms appear: $x^2=a$, $ax^2=b$, $x^2+a^2=b^2$.

The pupil's knowledge of geometry is used to rationalize several laws and processes of algebra, especially the laws used in multiplying algebraic monomials and polynomials. Thus emphasis in this unit is almost equally divided between algebra and geometry.

In the evaluation of polynomials, the solutions of equations, and checking of results ample practice is given in all the fundamental arithmetical processes. Further practice is offered in an abundance of practical applications and verbal problems.

Table XL exhibits the content of a unit on three-dimensional geometry. Formulas are developed for finding lateral areas, total areas, and volumes of the common solids. A careful study is made of each of the solids.

The first algebraic expressions of the third degree are found in the formulas for finding volumes. Such expressions as a^3, abc, πr^2h, and $\frac{4}{3}\pi r^3$ occur in numerous problems relating to the solids. Functional relationships are stressed and much practice in arithmetical manipulations is offered in the evaluation of algebraic expressions of the second and third degree.

With the unit on measurement of the solids the first stages of algebra and geometry are finished.

Correlation of mathematics and science.—Complaints are frequently voiced by teachers of science that pupils are not sufficiently grounded in the fundamentals of mathematics and that they do not know how to use the mathematics necessary to solve the problems that arise. There seems to be a lack of correlation between the two subjects, and one recommendation suggested as a solution of the difficulty is to bring about a closer union of mathematics and science.

The movement of correlating mathematics and science developed considerable force in the early years of the present century. The *Central Association of Science and Mathematics Teachers* was an outcome of this movement. One of its major objectives was to bring

TABLE XL*

Correlation of Mathematical Subjects in the Study of the Common Solids

Pupil Activities	Geometric Concepts	Geometric Principles	Algebraic Concepts	Algebraic Processes	Formulas; Equations	Relationships; Graphs	Arithmetical Processes
Making a model of the cube; making a drawing of the cube	Cube; faces; edges; vertices	....	....	....	....	....	....
Measuring the surface of the cube	Lateral area; total area	....	Polynomials of the second degree, as $ab+cd+ef$	Addition of quadratic terms; multiplication of linear terms	$t=6e^2$; solve $6e^2=80$	Between edge and surface of the cube	Square root
Measuring the content of a cube	Volume	A†	Monomials of the third degree, as e^3; polynomial of third degree, as x^3+2x^2+x+4	Evaluation of polynomials; multiplication of polynomials	Solve $v=e^3$ and $2x^3+10=26$	Between diagonal and edge of cube; between edge and volume of cube	Cube root by inspection; all processes are used in evaluation of polynomials
Making a model and drawing of a rectangular block	Rectangular block	B‡	Expression of the second and third degree, as in $t=2ab+2bc+2ca$, and $v=abc$	Evaluation of polynomials	$v=abc$	Between volume and dimensions of a rectangular block	All processes
Making a model and drawing of prisms; finding the volume	Prism; altitude; base	C§	....	Evaluation	$v=bh$	Between volume, base, and altitude of prism	All processes are used
Making models and drawings of cylinders; finding the volume	Cylinder; altitude; base	D‖; E¶	πr^2h	Evaluation; multiplication of polynomials by monomials	$L=2\pi rh$; $V=\pi r^2h$	Between lateral area, radius, and altitude; between volume and dimensions of cylinder	All processes; abbreviated multiplication and division

* A full discussion is found in *Eighth-Year Mathematics*, chap. iv.

† A. The volume of a cube is the cube of the edge.

‡ B. The volume of a rectangular block is the product of the three dimensions.

§ C. The volume of a prism is the product of the base by the altitude.

‖ D. The lateral area of a cylinder is 2π times the radius times the altitude.

¶ E. The volume of a cylinder is equal to π times the square of the radius times the altitude.

TABLE XL—*Continued*

Pupil Activities	Geometric Concepts	Geometric Principles	Algebraic Concepts	Algebraic Processes	Formulas; Equations	Relationships; Graphs	Arithmetical Processes
Making models and drawings of pyramids and cones; finding volumes	Pyramid; cone	F**; G††; H‡‡; I§§	$\frac{1}{3}\pi r^2 h$	Evaluation	$L=\frac{1}{2}ps$; $L=\pi rs$; $L=\frac{1}{3}bh$	Relation between lateral area, radius, and slant height; between volume, radius, and height of cone	All processes; abbreviated processes
Finding the area and volume of the sphere	Sphere; radius; diameter	J‖‖; K¶¶	$\frac{4}{3}\pi r^3$	Evaluation	$V=\frac{\pi r^2 h}{3}$; $S=4\pi r^2$; $V=\frac{4}{3}\pi r^3$	Between surface, volume, and radius of sphere	Abbreviated processes

** F. The lateral area of a pyramid is one-half the perimeter of the base by the slant height.

†† G. The lateral area of a cone is π times the radius times the slant height.

‡‡ H. The volume of a pyramid is one-third the base by the altitude.

§§ I. The volume of a cone is one-third π times the square of the radius times the altitude.

‖‖ J. The area of a sphere is 4π times the square of the radius.

¶¶ K. The volume of a sphere is $\frac{4}{3}\pi$ times the cube of the radius.

about a better correlation of mathematics and the other subjects of the curriculum, especially the sciences.

The movement was strongly indorsed by such leaders as Perry in England, Klein in Germany, and Moore in America. Progressive teachers of mathematics began to reorganize their courses, and reports of successful attempts to correlate mathematics and science soon appeared in the early issues of *School Science and Mathematics.* The laboratory method was a characteristic feature of the reorganized courses. Not only were mathematical principles established experimentally, but many experiments which traditionally belonged to courses in physics were performed in mathematics classes.

The correlation of mathematics and science is justifiable historically and pedagogically. Attempts to solve problems in the field of science have often made important contributions to mathematical knowledge. Indeed, much of the content and many principles of mathematics were actually discovered by men engaged in investigations in other fields. The invention of the calculus is an outstanding example.

From the standpoint of the study of mathematics much is to be gained by the correlation. Pupils are naturally interested in the practical uses of what they study. When mathematics is used by them in learning the principles of other subjects, they are impressed with the importance and value of mathematics in these subjects. They recognize the necessity of learning to use mathematics for purposes other than further study of mathematics. Indeed, research in physics cannot be carried on successfully without knowledge and understanding of mathematics. The physicist must know how to operate with mathematical symbols as much as with physical concepts. New theories are either verified by mathematics or discarded. Likewise, mathematics means much to the student of chemistry. Physics and chemistry are sometimes classified as branches of mathematics.

One of the immediate outcomes of the movement of correlating mathematics and science was the attempt to join the two fields by introducing in the study of algebra and geometry applications taken from physics and chemistry. However, it soon became apparent that these applications caused much difficulty to the pupils. This

should be expected unless the use of subject matter taken from the other courses is preceded by instruction in such subject matter. At least it should be carried on simultaneously with instruction in mathematics.

The problem of connecting mathematics with the natural sciences is thus related to the problem of preparation of teachers. Teachers of mathematics are not always familiar with the sciences taught in the secondary school and therefore not qualified to give the desired instruction. They can, however, co-operate with the teachers of science by paying special attention to the topics of mathematics in which pupils have been reported by teachers of science to show weakness. A list of the topics would have to be supplied by the science department.

The next step in the program may be to bring the two departments closer together by arranging the work of both so that mathematical processes be taught just before they will be needed in the sciences, or that topics of science be deferred until the necessary mathematical knowledge has been acquired. The plan should be worked out by a joint committee consisting of teachers from both departments.

Professor Moore advocated that mathematics and physics be taught jointly. This requires teachers familiar with the work of both departments. The experimental work may be done by the teacher of physics and the problem work may be directed by the teacher of mathematics. However, if it is possible the teaching of both subjects should be done by the same instructors. The program may be introduced gradually by having each year one teacher of science teach a course in mathematics and one teacher of mathematics teach one course in science. Frequent joint department meetings may be held until full co-operation is established.

Content and methods of the sciences to be stressed in mathematics.—Many useful and interesting facts of mathematics are often being withheld from the pupil until logical proof may be established. If they were derived experimentally they should be made available long before that time.

The experimental method should play a large part in all mathematical courses, especially in geometry. The first experiments should consist of simple measurements. Both the English and

metric system should be used. The measurements should be made in the classroom with the aid of ruler, compasses, squared paper, and protractor. The pupil should learn to measure line segments, angles, surfaces, and volumes. He should develop an understanding of the degree of accuracy to be attained in direct measurement and in deriving new facts from data obtained by measurement. Thus he should be able to compute areas and volumes from measures obtained by himself. He should learn to estimate, to compare magnitudes by means of ratio, and to find the average of several measures.

Many facts and laws of geometry and algebra should be discovered and formulated by the pupil on the basis of the findings of experiments. The sum of the angles of a triangle, the circumference of the circle in terms of the diameter, the congruence and similarity of triangles, the laws of signs, and the methods of solving equations illustrate this type of work. With a little apparatus some of the laws of physics may easily be developed in the geometry classroom, e.g., the laws of the parallelogram of forces, of leverages, of the inclined plane, of reflection and refraction of light, and of the composition and resolution of forces.

The ability to translate verbal statements into equations and formulas and to express the meanings of formulas in words is as important in science as in mathematics. The laws of physics will supply various types of equations that are usually studied in mathematics, and problems which lead to such equations. The use of these problems requires that the technical vocabulary of physics involving the terms "velocity," "force," "mass," "pressure," "Centigrade," "variation," and others be taught by teachers in both departments.

Finally, formulas and graphs are used as much in science as in mathematics to express relationships. They may be made the unifying factor of the two subjects. Science furnishes many helpful formulas by means of which practice may be given in calculating the value of one variable from known values of others, and in constructing graphs to make the relationships expressed by the formulas visible to the eye. Thus the relationship in the formula $pv=k$, which states that the volume of a gas varies inversely as the pressure, may be represented graphically by the hyperbola $xy=k$. The formula $F=\frac{9}{5}C+32$, which states the relationship between Centi-

grade and Fahrenheit, is represented graphically by a straight line. Other interesting formulas are $D=\frac{M}{V}$, which expresses the relationship between the density of a body, its mass, and its volume; $t=2\pi\sqrt{\frac{l}{g}}$, which shows how the number of vibrations of a pendulum varies with the length; $s=\frac{1}{2}gt^2$, which shows the relationship between the distance a particle has fallen and the time; and $p_1d_1=p_2d_2$, which expresses the law of leverages.

In view of the fact that science and mathematics have so much in common, a closer correlation between them should be of great value to both subjects. However, frequent experiences with pupils who do not know the mathematics needed in the study of the sciences have caused many teachers of science to become dissatisfied and to look upon mathematics as a disturbing factor in their subject. The complaint comes especially from teachers of physics. As a result the tendency has developed among some of them to dematthematize physics and to make the subject largely descriptive, with emphasis on knowledge of facts and subject matter.

This tendency has been greatly deplored as a false method of dealing with the problem by those who feel that physics is a study of relationships which are greatly simplified and made intelligible if formulated mathematically. They point to the findings of investigations of the mathematics needed in physics, such as the ability to carry out the processes and computations in physics problems, to recognize the mathematical concepts encountered in reading the textbook, to make accurate statements of the rules and theory of physics, and to do precise thinking about physical phenomena. Since the findings show that the mathematics actually used is not at all of difficult character, it is asserted that the fault lies with the mathematics teachers. If pupils would know the mathematics needed in physics, the subject would actually add to clearness and not to the difficulty of the study of physics. The solution of the problem is not the dematthematizing of physics but better teaching of mathematics.

To this criticism the teachers of mathematics reply that it is not based on facts, that they can present evidence that the pupils do know the mathematics, and that the real difficulty is lack of under-

standing of the physics situations. Since teachers of mathematics do not have the time or the equipment to teach these situations, the task is clearly one that belongs to the physics department.

Evidently neither department alone can solve the problem in a satisfactory manner. It seems to be a co-operative undertaking to which both departments must contribute. A close correlation between science and mathematics is the most logical step toward improvement.

BIBLIOGRAPHY

Barber, Harry C. "Improving America's Mathematics," *Mathematics Teacher*, XXV (May, 1932), 270–76.

Bass, Willard S. "An Attempt To Correlate Algebra and Physics," *School Science and Mathematics*, VI (June, 1906), 495–500.

———. "The Historical Argument for Teaching Arithmetic, Algebra and Geometry Together in the First Year of the High School," *ibid.*, V (December, 1905), 712–16.

Bishop, F. L. "Progress in the Correlation of Physics and Mathematics," *ibid.*, March, 1905, 152–59.

Blank, Laura. "The Influence of General Mathematics on the Subject Matter of Mathematics and on the Theory and Technique of the Teaching of Mathematics," *Mathematics Teacher*, XXI (October, 1928), 316–25.

Burks, W. D. "An Experiment Comparing the Efficiency of General Mathematics with Algebra and Geometry," *ibid.*, XVII (October, 1924), 343–49.

Breslich, E. R. *A Critical Examination and Experimental Reconstruction of Secondary Mathematics.* Doctor's dissertation, University of Chicago, 1926.

Cairns, W. D. "The Training of Teachers of Mathematics with Special Reference to the Relation of Mathematics to Modern Thought," *Mathematics Teacher*, XXIV (May, 1931), 269–76.

Carter, William R. "A Study of Certain Mathematical Abilities in High School Physics," *ibid.*, XXV (October, November, 1932), 313–31, 388–419.

Cobb, H. E. "Preliminary Report of the Committee of the Mathematics Section of the Central Association on the Unifying of Secondary Mathematics," *School Science and Mathematics*, VIII (November, 1908), 635–44.

Collier, Myrtle. *"The Need of a General Course in Mathematics," ibid.*, XXII (December, 1922), 845–49.

Crow, Jane M., and Dvořák, A. "A Study of Achievement in General Mathematics," *ibid.*, XXIX (January, 1929), 21–26.

Eells Walter C. "What Amount of Algebra Is Retained by College Freshmen?" *Mathematics Teacher*, XVIII (April, 1925), 219.

Evans, George W. "Some of Euclid's Algebra," *ibid.*, XX (March, 1927), 9–141.

Good, Carter V. "The Mathematics and Science Curricula in Junior and Senior High Schools," *School Science and Mathematics*, XXVII (November, 1927), 863–69.

Hedrick, E. R. "What Mathematics Means to the World," *Mathematics Teacher*, XXV (May, 1932), 249–63.

Karpinski, Louis C. "The Unity of Algebra and Geometry," *School Science and Mathematics*, XXXIII (May, 1933), 515–16.

Kilzer, L. R., and Kirby, T. J. *An Inventory Test for the Mathematics Needed in High-School Physics.* Bloomington, Ill.: Public School Publishing Co., 1929.

LeSourd, Homer W. "To What Extent Shall Secondary School Physics Be Mathematical?" *School Science and Mathematics*, XXXII (October, 1932), 777–84.

Lueck, William R. "How Much Arithmetic and Algebra Do Students of First Year College Physics Really Know?" *ibid.* (December, 1932), pp. 998–1005.

McCormick, Clarence. *The Teaching of General Mathematics in the Secondary Schools of the United States.* New York: Bureau of Publications, Teachers College, Columbia University, 1929.

Myers, G. W. "Outstanding Pedagogical Principles Now Functioning in High School Mathematics," *Mathematics Teacher*, XIV (February, 1921), 57–63.

Newell, Charles. "Correlation of Mathematical Studies in Secondary Schools," *Proceedings of the National Education Association* (1902).

Pierce, Paul R. "Report of an Experiment in Correlated Mathematics in a Large High School," *School Science and Mathematics*, XXV (October, 1925), 681–84.

Reagan, G. W. "The Mathematics Involved in Solving High School Physics Problems," *ibid.*, XXXV (March, 1925), 292–99.

Rendahl, J. Z. "The Mathematics Used in Solving Problems in High School Chemistry," *ibid.*, XXX (June, 1930), 683–89.

Rich, Ashley L. "An Argument for a Correlated Course in Science and Algebra," *Mathematics Teacher*, XXV (January, 1932), 33–35.

Sanford, Vera. "Textbooks in Unified Mathematics for College Freshmen," *ibid.*, XVI (April, 1923), 206–14.

Smith, D. E. "Mathematics in the Training for Citizenship," *Teachers College Record*, XVIII (May, 1917), 211–25.

Stone, Charles A. "Correlation of the Mathematical Subjects Develops Mechanical Power," *Mathematics Teacher*, XVI (May, 1923), 302–10.

Straley, H. W. "The Deficiency in Mathematical Training Required of Geologists," *School Science and Mathematics*, XXXII (October, 1932), 745–47.

Tripp, M. O. "Applications of Indeterminate Equations to Geometry," *Mathematics Teacher*, XXI (May, 1928), 268–72.

Wallace, Raymond R. "The Relative Values of Unified and Correlated Mathematics in Presenting the Fundamental Operations," *School Science and Mathematics*, XXVIII (October, 1928), 740–45.

Williams, L. W. "The Mathematics Needed in Freshman Chemistry," *ibid.*, XXI (October, 1921), 654–55.

Webber, W. Paul. "Combined Mathematics," *Mathematics Teacher*, XIV (November, 1921), 381–86.

Zerbe, Hobson M. "The Elements of Plane Geometry in High School Physics," *School Science and Mathematics*, XXX (June, 1930), 665–67.

———. "The Elements of Plane Geometry in Plane Trigonometry," *ibid.*, December, 1930, p. 1020.

CHAPTER X

PLANNING THE TEACHING OF A BODY OF INSTRUCTIONAL MATERIALS

Meaning of the term "unit."—To many teachers the literature relating to the units of secondary-school mathematics has been greatly confusing because of the variety of meanings assigned to the term "unit." To some writers it designates the lesson for the day; to others it is a block of work, as a chapter in a textbook, a project, or a job; and still others speak of it as a method of instruction, rather than of organization of instructional materials.

If the unit is the lesson for the day, the plan for teaching is simple. It aims to help the pupil attain mastery of the exercises, propositions, and processes in that lesson. The goal is the ability to recite the assignment in a satisfactory way. Thus, in algebra the unit may mean a process to be studied intensively by itself to the exclusion of all others. In geometry it may mean a theorem to be mastered without reference to the others.

Writers of textbooks and curriculum-makers have always found it convenient and helpful to organize instructional materials into blocks or chapters. In the first textbook on geometry, written by Euclid, the subject was divided into "books," which may be regarded as his units of instruction. He made "logical sequence" the basis of arrangement of the theorems and problems of geometry. No account was taken of the relative difficulty of the theorems. Hence, no attempt was made to teach first the simple principles and later the more difficult. Euclid was not concerned about grouping the propositions in a way in which they may be most conveniently and most easily taught and learned, but arranged them to follow each other in the order demanded by his logical system. This was entirely suitable to his purposes and needs since in his time the students of geometry were more interested in the logic of geometry than in other phases of the subject. Hence, it was natural to organize the instructional materials logically. Moreover, the logical organization was so thoroughly carried out by Euclid that it has stood

the test of over two thousand years of use. Indeed, with slight modifications this is the most commonly accepted classification of today. Many textbooks in plane geometry are still organized according to the book system of Euclid, and some writers still refer to these books as the "units of instruction."[1]

The same conception of the "unit" of geometry as a body of materials organized on the logical basis is expressed more recently by some writers who look upon the unit as a collection of theorems and exercises to be learned within a specified period of time. The *Fifth Yearbook* of the National Council of Teachers of Mathematics contains two articles on "The Unit of Demonstrative Geometry for the Ninth Year." One of the writers speaks of the unit as a "series of exercises which are based upon certain postulated facts and which lend themselves to logical demonstration" to "cover in time the equivalent of six to eight weeks" to be fit into the work of the ninth year. He organizes the "unit" into topics, such as preliminary definitions; vertical angles are equal; the congruence theorems; parallel lines; and similar triangles.[2] The other writer interprets the term "unit" in demonstrative geometry for the ninth school year to mean "the demonstration of a limited number of propositions . . . . the principal purpose being to show the pupil what *demonstration* means . . . . a sequence of propositions expressly so chosen as to make the introduction to logical proof most real, most palatable and least painful to the average pupil."[3]

The disadvantages of making logic the only basis of organization of geometric materials for high-school teaching are evident. The daily lessons necessarily vary widely in difficulty. Some days the work is too simple; on others it is discouragingly difficult. If the lessons are related in other ways than in logical sequence, that fact will not be clear to the learner. He finds it exceedingly difficult to retain unrelated facts. Each new proposition is taught and

[1] S. C. Sumner, *Supervised Study in Mathematics and Science* (New York: Macmillan Co., 1922), p. 105.

[2] Joseph B. Orleans, "A Unit of Demonstrative Geometry for the Ninth Year," *Fifth Yearbook* (New York: Columbia University Press, 1930), pp. 44–53.

[3] Joseph Seidlin, "A Unit of Demonstrative Geometry for the Ninth Year," *ibid.*, pp. 54–63.

learned by itself, to be followed by practice, applications, and the recitation. As the pupil continues in the study, he tries to escape confusion and failure by memorizing the propositions. Thus, he falls into wrong habits of study which may ultimately defeat the real aims to be accomplished by the course. He fails to comprehend the broad and most valuable principles of geometry. This is contrary to the modern tendency in teaching geometry, which is away from learning merely the finished proofs of others and in the direction of developing power to solve original exercises.

Traditionally the courses in algebra are organized in chapters which teach the pupil to add, to subtract, to multiply, to divide, to manipulate signed mnubers, to solve equations, to solve problems, and to make graphs. These chapters have been thought of as the units of algebra. Each is taught and studied by itself. Each may go far beyond the present or remote needs of the pupil. The learner does not see the relationships between chapters. Nor does he see how each chapter contributes to the course as a whole. His only aim is to finish the chapter and to have it over with. As he goes on with the course he continues to develop the attitude of getting through. When he solves equations, he does not want to be bothered with factoring, and when he works with fractions, he does not want to be interrupted with equations. While he studies addition he adds anything from the simplest problem to one so difficult that it is found nowhere in the field of mathematics. He does not know what all the complicated work in adding leads to, and he does not care to know.

In the last thirty years the leaders in mathematics have searched for broad unifying principles which could be made the core of the mathematical courses. The most frequently mentioned unifying factor in arithmetic, algebra, geometry, and indeed in all mathematical subjects is the function concept. If unifying principles are made the bases of organization, the courses will be greatly improved. Suppose that complete understanding of the function $y=ax+b$ be made one of the major divisions of first-year algebra. The attainment of an understanding of this important function will depend upon many experiences. For example, the pupil must be able to find the value of $ax+b$ for given values of a, x, and b. He must therefore learn to make substitutions. He must acquire a knowl-

edge of the laws of multiplication, addition, and subtraction for positive and negative numbers. He may tabulate the results and observe the changes of the value of $ax+b$ corresponding to the changing value of x. A graph of $ax+b$ will make the changes even clearer, for the graph readily shows the value of $ax+b$ for any particular value of x. It gives a concrete meaning to the coefficients a and b. It also answers the question of the values of x for given values of $ax+b$, including the value zero. This leads directly to the study of linear equations. Since $x=(y-b)/a$, some knowledge of fractions may be attained. Finally, it is possible to make up verbal problems leading to $y=ax+b$, and some work in problem-solving may be offered. Thus, it is evident that to attain complete understanding of the function $y=ax+b$ the pupil must learn the operations with positive and negative whole numbers and simple fractions. He must know graphical representation. He must be able to solve simple equations, and problems leading to such equations. Complete understanding of the function $ax+b$ may therefore be regarded as a unit of the course.

The illustration shows the advantage of organizing a body of materials on the basis of unifying principles. Such an organization enables the pupil to see clearly the relationship between the various facts, processes and principles taught in the course. He knows what each unit contributes to the course as a whole. If one adds to the study of the linear function $y=ax+b$ that of the quadratic function $y=ax^2+bx+c$, it will be possible to construct a course which is equivalent in content to the traditional algebra and much simpler to assimilate.

The question will be asked as to how the teacher who must use a textbook organized in chapters may secure the advantages of unitary organization. A chapter is not necessarily a unit but it may be possible to change it to a unit by reorganizing it on the basis of broad principles. To illustrate, in his college mathematics the writer had the good fortune of working under an instructor whose first assignment on a new chapter was always the entire chapter. To the students this seemed to be an unreasonable assignment, but it turned out to be of great value to them. Having examined the chapter as a whole, they knew from the first day on to the end of the chapter what it was all about. They saw how the work of each

day will contribute to the large principles of the chapter. They recognized the mutual relationships of the instructional materials. They were in a position to study the chapter more intelligently than would have been the case if each day's work had been assigned and learned by itself. As a result all enjoyed the course and knew a great deal about it when it was finished. For example, the textbook contained a chapter on the solution of equations. It gave theorems on the roots of an equation, rules of signs, graphical work, and ways of transforming equations. However, each theorem, rule and process was presented by itself without reference to the orders. The discussion and recitation following the assignment of the whole chapter brought the theorems, rules, and processes into relation to one another. It was shown that the aim of the chapter was to enable the student to find the roots of rational integral equations of nth degree; that the roots of such equations may be real or complex; if real, that they may be rational or irrational; if rational, that they may be integral or fractional; and that some of the principles would enable the student to find the integral roots, others to find fractional roots. As the discussion went on, the students were led to see that each theorem had a definite purpose and therefore needed to be thoroughly understood. They saw at the beginning what usually remained hidden to them until the study of the chapter was finished.

In teaching high-school pupils the instructor might have gone a little farther. He could have started by giving the student a clear view of the chapter. This preview might have stressed and illustrated the following points: "The chapter teaches you to solve equations like $3x^4-x^3+4x^2+x-8=0$ which shall be called rational integral equations. The roots of such equations are either real or complex. If real, they may be rational or irrational. If rational, they may be integral or fractional. You will first learn how to find the integral roots of an equation, and then take up the other types in the order indicated. The first problem calls for the use of certain principles which must be understood. You will find them in your book on pages. . . . . Your first assignment will be to acquaint yourselves with these principles and to learn to use them."

The instructor could have illustrated each of the various statements of the preview by concrete examples to show what was meant. In the study of the chapter this would help the pupils to go

about their work intelligently, understanding the connections between the various principles and seeing the relationships of the theorems to the chapter as a whole. Thus the chapter would have been transformed into a unit.

The question of time and size need not enter in the organization of a unit. However, it is difficult for most high-school pupils to form a clear conception of a body of materials which it would take more than a month to assimilate. Experience seems to show that a unit which can be studied in three to four weeks is most suitable as to size. When it is not possible to finish it within that time, the teacher should find a way of simplifying the unit either by transferring some of the materials to other units or by dividing it into two smaller units. For example, the materials relating to the study of the circle in plane geometry form too large a unit and should therefore be reclassified to form smaller units like the following: the relationships between chords, arcs, and central angles; measurement of angles by circle arcs; relationships between segments of intersecting chords, secants, and tangents; and the regular polygons inscribed in and circumscribed about the circle.

Another example of searching for a satisfactory organization of materials by classifying subject matter on the basis of broad unifying principles is reported by Schultze.[4] A number of years ago he called attention to the fact that the theorems illustrated in Figure 21 are really different aspects of the principle that *an angle is*

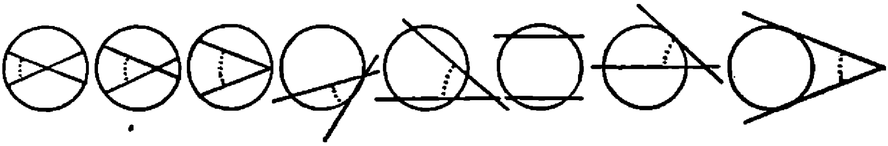


FIG. 21

measured by one-half of the algebraic sum of the intercepted arcs. The pupil who studies the theorems as special cases of one principle will learn the separate theorems and in addition he will see that they are related to the general unifying principle and that one merges into the other when the two intersecting lines are moved about. This group of propositions may therefore be organized into a unit

[4] Arthur Schultze, *The Teaching of Mathematics in Secondary Schools* (New York: Macmillan Co., 1927), p. 185.

in the sense in which that term is used in this discussion. They may be presented and studied as a group rather than as separate facts.

Likewise, other groups of propositions may be found in Euclid's book on rectilinear figures. The theorems on congruence, similarity, and proportionality are typical illustrations. Not only mastery of the separate theorems on congruent and similar figures, but the development of power to use the ideas of congruence and similarity in solving original exercises are the objectives of these units.

The foregoing examples illustrate that unitary organization has the following characteristics:

1. It organizes a body of facts, theorems, or processes, closely related to one another and so organized as to contribute to the understanding of an important aspect of the course. In the foregoing illustration of measurement of angles by arcs the unifying principle is the law by which an angle may be measured by the intercepted arcs of a circle.

2. It must be possible to present the theorems and processes as a group in a form so definite that the learner may attain a conception of them before he undertakes the detailed study of content of the unit. In the illustration a view of the group of theorems may be presented in a few minutes in a brief talk void of technical terms in a way which will show the pupil exactly what the unit aims to accomplish and the way he is going to travel as he studies it.

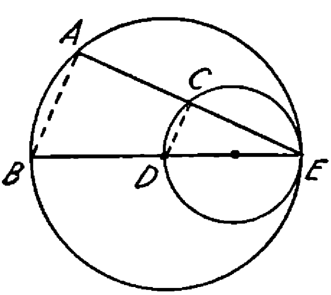


FIG. 22

3. It must be possible to set up outcomes of the study so definite that they are clear not only to the teacher but also to the pupils. The ends to be attained in the illustration are: Complete understanding of the general relationships between angles and circle arcs and of the special cases that may be formed, and the power to recognize and use the various theorems when they arise in problem situations. For example, the pupil examining Figure 22 should be able to recognize that the measure of each of the angles *BAC* and *DCE* is 90, since they are measured by one-half of the semicircles. This fact makes the angles equal and the lines *AB* and *CD* parallel.

The foregoing discussion has shown a variety of meanings attached by various writers to the term "unit." By way of illustrations it has been attempted to explain the meaning of the term as it is used in this volume. It should be clear that there is no single type of organization which may be set up to determine an "only acceptable" list of units for a course. Units are organized to attain certain objectives of the teaching of mathematics. They must be subjected to experimentation and revision. A different set of objectives might influence considerably the choice of units. It is not intended to urge teachers and supervisors to accept any particular list of units but rather to illustrate ways by which they may experiment with organizations that will be suitable to their own needs and tastes.

Unitary organization makes teaching and learning purposeful and intelligent. Because the instructional materials are closely related to each other they are easily retained. Economy of time and effort should be the result.

The task of selecting the instructional materials calls for keen judgment and broad understanding of the teacher. Most teachers will use as far as possible the materials found in the adopted textbook. They will select from other progressive books supplementary materials that promise to be helpful in the study of a unit. The present chapter aims to illustrate with a particular example a plan which may be used in the teaching and study of a given unit.

Euclid's Book I, on rectilinear figures, contains discussions of parallel lines, perpendicular lines, triangles, quadrilaterals, polygons, and loci. These materials may be reorganized into a number of pedagogical units. The first deals with the facts relating to parallel and perpendicular lines and the second with the group of propositions relating to the properties of such quadrilaterals, as parallelograms, rectangles, squares, and trapezoids. The second has been selected to illustrate the plan of organizing a unit for teaching purposes.

Setting up objectives to be attained.—Attention has been called previously (chap. iii) to the importance and value of listing definite objectives in teaching and in study. An important step in the plan of organizing a unit should be to decide upon the unit objectives to be attained. For the unit on quadrilaterals eight objectives have

been chosen which the teacher should keep in mind constantly as the class studies the unit. They are:

1. A knowledge of the names and meanings of various quadrilaterals, e.g., the parallelogram, rectangle, square, trapezoid, and rhombus.
2. An understanding of the relations of the different quadrilaterals to each other and to the general quadrilateral. Thus, a rectangle is a parallelogram having right angles; a rhombus is an equilateral parallelogram; and a square is an equilateral rectangle.
3. Skill in constructing parallelograms with ruler and compasses, and with squared paper.
4. Understanding of the properties of the quadrilaterals studied in the unit, e.g, the facts that the opposite sides of a parallelogram are equal; and that the diagonals bisect each other.
5. A knowledge of the conditions under which a quadrilateral is a parallelogram, rectangle, square, or rhombus.
6. Acquisition of power to solve original exercises by the most commonly used methods of proof, such as the analytic method, the algebraic method, and the congruent-triangle method.
7. Development of power to read understandingly the finished proofs given in the textbook.
8. Appreciation of the importance and value of the unit in practical applications and in various school subjects.

The exploration.—Most pupils who come to the study of this unit have had many geometric experiences. All know something about the materials to be presented and some know a great deal about them. The teacher who makes pupils go over subject matter with which they are familiar will find them restless, uninterested, and often openly antagonistic. It is necessary, therefore, to determine at the outset of this unit what they know about quadrilaterals. One method commonly employed by teachers is that of obtaining the desired information by carefully planned questions and by drawing the pupils into discussions. Questions like the following show how this may be done: Do you see any four-sided figures in the classroom? Have you noticed four-sided figures out of doors? Can you give the names of them? What are the essential characteristics of the rectangle and of the general parallelogram? Do you know any practical device whose construction is based on

the parallelogram? Do you recall instances in which you made use of your knowledge of quadrilaterals? Try to find one in physics, one in science, and one in physical education.

The foregoing method of exploration is easy to administer but the disadvantages are evident. The teacher is likely to lose time attempting to draw from the students information which they do not possess. Although the knowledge of the class as a whole may be revealed in a general way, it is difficult to identify the specific knowledge of individuals. There is danger of being sidetracked in discussions and of mixing a certain amount of aimless teaching with the exploration.

A more effective method of exploration is to make use of a pretest prepared to obtain an inventory of the pupil's knowledge of the unit. To be comprehensive such a test must make use of the modern testing devices. A pretest for the unit on quadrilaterals follows.

PRETEST FOR THE UNIT ON QUADRILATERALS

1. On the line under each figure write the name which describes the figure best (Fig. 23).

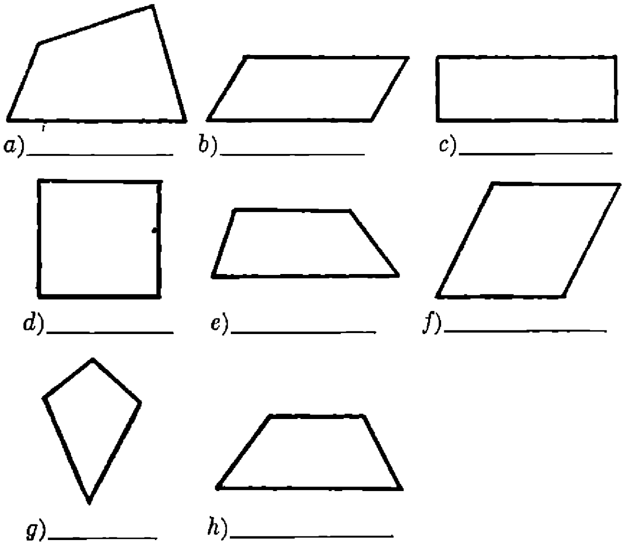
Figures	Names of Quadrilaterals
	Isosceles trapezoid Rectangle Square Rhombus Quadrilateral Parallelogram Trapezoid Kite

FIG. 23

2. On the blank lines write the words that make the sentences true:
   a) A diagonal divides a parallelogram into two ....
   b) The .... sides of a parallelogram are equal.
   c) The .... angles of a parallelogram are equal.
   d) Parallel segments .... parallels are equal.
   e) The diagonals of a parallelogram ....
   f) The .... angles of an isosceles trapezoid are ....

3. Complete the following statement:
   ABCD (Fig. 24) is a parallelogram because
   (1) .... is constructed
   ....
   (2) .... is constructed
   ....

FIG. 24

4. The wall table shown in Figure 25 may be moved to and from the wall. State a geometric theorem which causes it always to remain in horizontal position.
   Answer: It stays in horizontal position because..
   ....
   ....
   ....

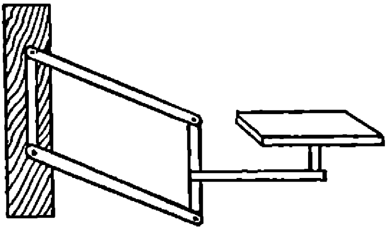


FIG. 25

5. *ABCD* is a parallelogram. On the blank lines state in symbols the relationships between:
   a) *AB* and *DC*
   b) *AE* and *EC*
   c) $\angle BAD$ and $\angle ABC$
   d) $\angle BAD$ and $\angle BCD$ ....
   e) $\angle ABC$, $\angle BCD$, $\angle CDA$, $\angle DAB$ ....
   f) $\triangle AEB$ and $\triangle DEC$

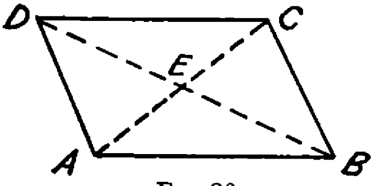


FIG. 26

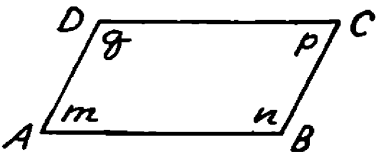


FIG. 27

6. Let $m=p$ and $n=q$ (Fig. 27). Give the reason for each of the following statements:

Statements:	*Reasons:*
(1) $m+n+p+q=360$	....
(2) $m+n+m+n=360$	....
(3) $2m+2n=360$	
(4) $m+n=180$	
(5) $AD \parallel BC$	....

7. If $AB=DC$ and $AB \parallel DC$ (Fig. 28), you are to prove that $ABCD$ is a parallelogram. The statements for the proof are given below, but they are not arranged in logical order. You are to arrange them mentally in ordered logical sequence and to write the numbers of the steps in the parentheses.

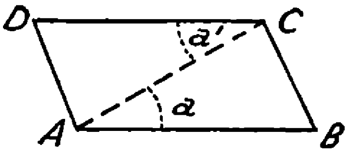


FIG. 28

Statements:	Logical Ordered Sequence
(1) $\triangle ABC \cong \triangle ADC$	Statement (3) should come first.
(2) $AD=BC$	Statement ( ) should be second.
(3) $AB=DC$	Statement ( ) is the third.
(4) Draw AC	Statement ( ) is the fourth.
(5) $a=a'$	Statement ( ) is the fifth.
(6) $ABCD$ is a parallelogram.	Statement ( ) is the sixth.
(7) $AC \equiv AC$	Statement ( ) is the seventh.
(8) $AB \parallel DC$	Statement ( ) is the eighth.

8. $ABCD$ is a parallelogram. If AB is denoted by $3x+2$ and DC by $5x-28$, find the value of x.

The responses of twenty-seven pupils to the foregoing inventory test have been arranged on a diagnostic record sheet in Table XLI. It is evident that they knew a great deal about quadrilaterals, especially about the informational phases. The explanation is that the majority of the pupils in this group had studied intuitive geometry in the junior high school. Without the knowledge of the results disclosed by the test, valuable time would have been spent unnecessarily in the study of the unit, since the teacher would probably have taught the facts to be tested by items 1_b, 1_d, 6_3, and 6_4 with the same care as 1_g or 2_f. Furthermore, Pupils 3, 9, and 20 understood about two-thirds of the items given in the test. With-

out studying the unit, they were able to do what was required and needed merely some additional work.

A careful analysis of the results of the inventory test will make the teaching of a unit more intelligent than would be possible without the information which it contains.

TABLE XLI

INVENTORY TEST ON QUADRILATERALS

Test Items / Pupils	1								2						3	4	5						6					7	8	Total Right
	a	b	c	d	e	f	g	h	a	b	c	d	e	f			a	b	c	d	e	f	1	2	3	4	5			
1	X		X		X	X	X	X			X	X	X	X	X		X	X	X	X		X					X	X	X	10
2	X				X	X	X	X			X	X	X	X	X	X	X	X	X	X		X	X				X	X	X	9
3							X					X	X	X		X														24
4	X				X	X	X	X	X	X	X	X	X	X	X	X	X	X		X	X	X	X	X			X	X	X	6
5					X	X	X	X			X	X	X	X	X													X		19
6	X		X			X	X		X		X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		X	X	6
7	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		X		X			X	X	4
8	X	X	X		X	X	X	X	X		X			X			X											X		17
9	X						X	X				X	X	X																23
10						X	X				X	X	X	X							X		X					X		20
11	X		X		X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		X	X	X	X	3
12	X					X					X	X	X	X		X			X	X	X		X				X	X	X	15
13						X	X	X	X		X	X	X	X	X	X		X	X	X	X		X				X	X	X	11
14	X	X			X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		X	X		X	X	X	X	4
15	X		X		X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		X	X		X	X	X	X	4
16	X		X			X	X		X		X	X	X	X			X				X		X					X		16
17	X				X		X	X	X	X	X			X		X	X													19
18	X				X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		X	X	X	4
19	X					X	X				X	X	X	X		X	X			X	X		X				X	X	X	14
20					X		X	X				X	X	X									X							22
21							X		X			X	X	X							X		X					X		21
22	X	X			X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		X	X	X	3
23						X	X	X			X	X	X	X						X			X					X		19
24						X		X			X	X	X	X	X		X	X		X							X	X	X	16
25	X					X	X					X	X	X	X	X		X	X		X		X				X	X	X	14
26	X	X	X		X	X	X	X	X	X	X	X	X	X	X	X	X	X			X	X	X				X	X	X	6
27	X					X	X		X		X		X	X			X	X	X	X	X	X	X				X	X	X	12
Total	8	22	19	26	13	5	2	9	12	18	5	3	2	0	12	11	11	11	14	11	10	18	7	20	23	23	12	4	10	341

The preview of the unit.—The purpose of the preview of the unit is to let the pupil view it as a whole before he begins to study details and to call his attention to the relations which exist among the various facts and principles that make up the unit. Some writers strongly advocate the preview as a necessary part of the plan of teaching the unit. Others disapprove of it on the ground that it makes the learner dependent on the teacher and that he is being deprived of the opportunity to form his own view of the unit. The advantage of obtaining a clear conception of the nature of the unit before the study is undertaken must be recognized. However, the

methods of obtaining it may vary with different units and for different levels. When experience has shown that pupils find it too difficult to form a conception of a given unit independently, the preview will be desirable. Otherwise, the task may be assigned to them and may be carried out independently or with the teacher's guidance. In all cases the teacher may determine by means of a written exercise whether the pupils have actually acquired a view of the unit.

A preview given by a teacher for the unit on quadrilaterals is reproduced below in abbreviated form.

The central theme of the unit on quadrilaterals is the study of the properties of certain well-known four-sided figures. I am making a diagram representing the most general four-sided figure. It is called a "quadrilateral" (Fig. 29).

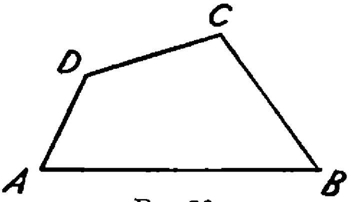


FIG. 29

If DC is turned about point D, as shown by a pointer which I am placing along DC and which I am turning, the shape of the quadrilateral changes. When the pointer, or DC, reaches the position in which it is parallel to AB, the quadrilateral is called a "trapezoid" (Fig. 30). Thus *a trapezoid is a quadrilateral two of whose sides are parallel to each other.*

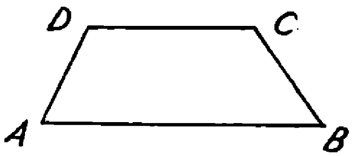


FIG. 30

Now let me change the figure again. I shall place the pointer along CB. Keeping C fixed, I am turning CB, making point B move along BA until CB is equal in length to DA. The trapezoid is now called "isosceles" (Fig. 31). Later you will be asked to prove that *the base angles, A and B, of an isosceles trapezoid are equal.*

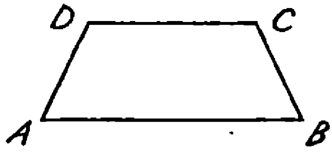


FIG. 31

I shall make another change by turning CB about C until it is parallel to DA. The inventory test shows that practically all of you know the name of this figure. It is a "parallelogram." Thus, *a parallelogram is a quadrilateral with two pairs of parallel sides.* A large part of this unit is concerned with the properties of the parallelogram. Thus, it will be proved that the opposite sides are equal; that the opposite angles are equal; that the consecutive angles are supplementary; and that the diagonals bisect each

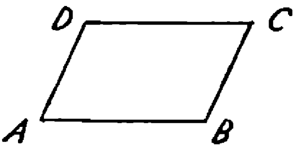


FIG. 32

other. The converse theorems will also be proved because they state the conditions under which a quadrilateral is a parallelogram.

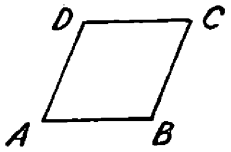


FIG. 33

Now imagine that side BC of the parallelogram be moved, always remaining parallel to its first position, until AB is equal to AD. The new figure formed is an equilateral parallelogram. It is called "rhombus" (Fig. 33).

All the theorems proved for the parallelogram will hold for the rhombus because it is a special kind of parallelogram. However, not all the facts that are true for the rhombus apply to the general parallelogram. For example, the diagonals of a rhombus are perpendicular to each other, but this is not true for all other parallelograms.

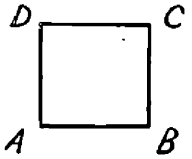


FIG. 34

If sides AD and BC of the rhombus be turned about A and B, respectively, until angles A and B are right angles, the figure formed is a "square" (Fig. 34). Thus, the facts proved for the rhombus will hold for the square.

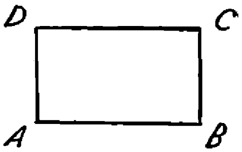


FIG. 35

Now let us change angles A and B of the parallelogram (Fig. 32) until they are right angles. The parallelogram is then changed into a "rectangle" (Fig. 35). A rectangle is therefore a special kind of parallelogram. This fact will enable you to state some interesting properties of the rectangle.

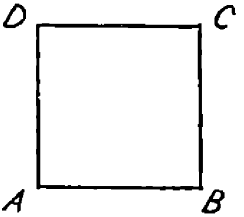


FIG. 36

If side DC of the rectangle (Fig. 35) is moved parallel to its original position until AD is equal to AB, a "square" is formed (Fig. 36). Hence a square is a special case of a rectangle as well as a special case of a rhombus.

It is of interest that several of the principles of the quadrilaterals studied in the unit have practical value. Thus, the fact that the ironing board (Fig. 37) shown in the diagram always remains in horizontal position as it is raised or lowered is based upon one of the properties of parallelograms that will be proved in this unit.

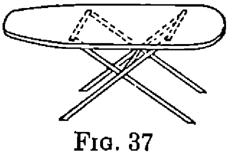


FIG. 37

Further uses of the theorems taught in this unit will be found in constructions of parallelograms some of whose parts are given.

The unit will introduce exercises to enable you to use the theorems proved in establishing new facts. Another purpose of the exercises is to give training in correct reasoning and in solving originals.

The study outline of the unit.—The purpose of the outline is to provide the pupil when he begins the study of the unit with directions as to what to study and how to do it most effectively. The following outline has been used in the unit on quadrilaterals. It refers to the basic text, Breslich, *Senior Mathematics*, Book II. References to other texts may be added by the teacher. Furthermore, he may include references to discussions or illustrations of the principles studied in the unit. Mimeographed copies should be made and each pupil should receive a copy.

STUDY OUTLINE

The study of the unit on quadrilaterals has been divided into three parts.

PART I. PROPERTIES OF PARALLELOGRAMS

Study carefully *Senior Mathematics*, II, 66–68. You should accomplish the following:

A. You should be able to give correct meanings of the terms "quadrilateral," "parallelogram," "diagonal," "opposite angles of parallelogram," and "consecutive angles of parallelogram."

B. You should understand the proofs given in §§ 68 and 69, and should be able to reproduce them without referring to the book.

In the study and review of the proofs given in the textbook the following suggestions will be helpful to you:

Be sure that you understand the meaning of each term in the theorem.

Be able to state the theorem fluently.

Know what is given and what is to be proved.

Be able to recall the diagram, especially the helping lines drawn in it.

Remember the method of proof that is used.

Do not memorize the steps of the proof but fix in mind an outline of the major parts into which the proof is divided.

Thus, before you study the proof given in § 69 you should understand the meanings of the terms "diagonal," "parallelogram," and "bisect." You should then fix in mind the method of proof, i.e., the method of congruent triangles; the helping lines drawn in the diagrams; and the essential steps of the proof, i.e., (1) that two triangles are congruent and (2) that the corresponding sides of congruent triangles are equal.

Note down the statements in the proof that you cannot understand. They will be explained to you by the instructor.

C. Write into your notebook the proofs of Exs. 1–4, p. 67. Be prepared to prove the exercise on p. 68.

D. When you have finished Part I you may verify your understanding by means of the following test. Do not take the test until you feel sure that you are thoroughly prepared.

1. A quadrilateral is a .... polygon.
2. A diagonal is a line joining two .... not on the same ....
3. The opposite .... of a .... are .... and the opposite .... of a .... are ....
4. In the parallelogram $ABCD$ (Fig. 38), $m=n$ because if two parallel lines ....

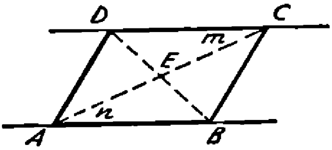


FIG. 38

5. $\triangle ABC \cong \triangle ADC$ because two triangles are congruent ....
6. $AD=CB$ because parallel segments .... parallels ....
7. $\triangle AEB \cong \triangle DEC$ because two triangles are congruent if..
8. $\angle ADC = \angle ABC$ because the corresponding angles ....
9. $AE=EC$ because the corresponding sides..

If you are unable to complete some of the statements in the test, continue to study Part I or ask your instructor for assistance.

Part II. Conditions under Which a Quadrilateral Is a Parallelogram

A. Study *Senior Mathematics*, II, 69–77. Recall and use the suggestions given for Part I.

B. Keep a careful record of any facts that you do not understand. They will be explained by the instructor. Study the four theorems in Part II and practice until you are able to reproduce each proof without the use of the textbook.

C. Be able to construct a parallelogram as shown in Ex. 1, p. 70.

D. Exs. 2, 7, and 8, p. 70; Ex. 4, p. 72; Ex. 3, p. 74; and Exs. 5 and 6, p. 76, involve solutions of linear equations. You may have to review the method of solution. Be prepared to explain it to the class.

E. Find out which theorems apply in the practical problems in Exs. 3 and 4, p. 70, and in Ex. 2, p. 74.

F. Search current magazines, textbooks, encyclopedias, and other books in the library for applications of the theorems taught in this unit and write them in your notebook.

G. Be prepared to discuss the solutions of Exs. 5, 6, p. 70; Exs. 1, 2, 3, 5, 6, 7, p. 72; Exs. 1, 4, 5, p. 74; and Exs. 1, 2, p. 76. Write the more difficult proofs into your notebook. The purpose of the exercises is to give you training in solving originals and in drawing logical conclusions. Exercises too difficult for you will be explained in class. The remaining exercises you are to solve without assistance.

H. Ex. 7, p. 76, will be explained by the teacher if help is needed. You are then to work out Exs. 8 and 9 in a similar way.

I. Review and make a summary of what you have learned in Parts I and II.

J. When you have finished Part II you may test your knowledge by means of the following items. Be sure not to take the test before you feel thoroughly prepared.

1. *ABCD* (Fig. 39) is a parallelogram:

if $\begin{cases} AB = \ldots\ldots \\ AB \parallel \ldots\ldots \end{cases}$

or if $\begin{cases} AB = \ldots\ldots \\ AD = \ldots\ldots \end{cases}$

or if $\begin{cases} AE = \ldots\ldots \\ EB = \ldots\ldots \end{cases}$

or if $\begin{cases} \angle BAD = \ldots\ldots \\ \angle ADC = \ldots\ldots \end{cases}$

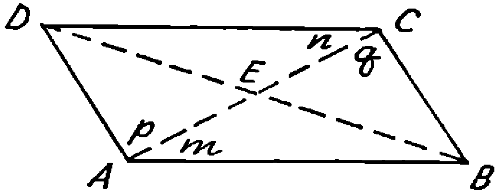


FIG. 39

2. Let $EF = HG$ and $EH = FG$ (Fig. 40). To prove *EFGH* a parallelogram I prove

$\triangle \ldots\ldots \cong \triangle \ldots\ldots$

$\angle \ldots\ldots = \angle \ldots\ldots$

$\ldots\ldots \parallel \ldots\ldots$

$\angle \ldots\ldots = \angle \ldots\ldots$

$\therefore \ldots\ldots \parallel \ldots\ldots$

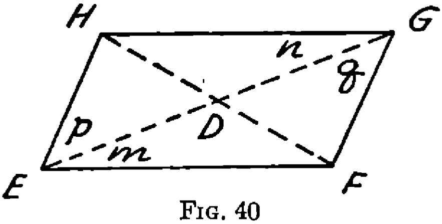


FIG. 40

3. Let $AB = DC$ and $AB \parallel DC$ (Fig. 41). To prove *ABCD* a parallelogram, I draw line ....

Then I prove $\triangle \ldots\ldots \cong \triangle \ldots\ldots$, using the fact that two triangles are congruent if...

It follows that .... = ...

Hence *ABCD* is a parallelogram because ....

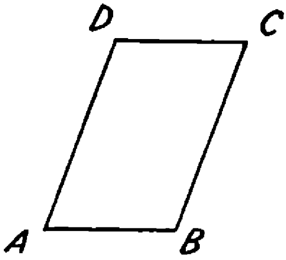


FIG. 41

4. Let $PT=TR$ and $QT=TS$ (Fig. 42). To prove $PQRS$ a parallelogram, I prove

$\triangle$............ $\cong$ $\triangle$............
$\therefore$ .... $=$
and .... $=$
$\therefore$ .... $\parallel$ ....
$\therefore$ $PQRS$ is a parallelogram because............................
....

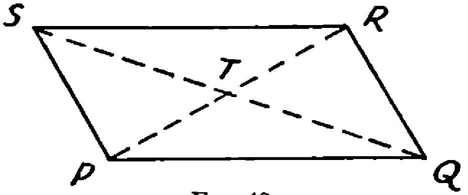


FIG. 42

5. If $m=q$ and $p=n$ (Fig. 43), I can prove that the quadrilateral $MNST$ is a parallelogram by proving
.... $=360$
.... $=180$
$\therefore$ ....
Similarly, ...

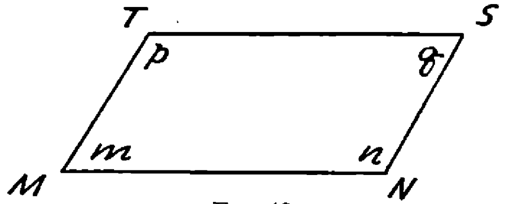


FIG. 43

6. To construct a parallelogram, I draw AB and AC (Fig. 44). Using .... as center and .... as radius, I draw an arc at D. Using .... as center and .... as radius, I draw another arc intersecting the first at .... I then draw .... and ....

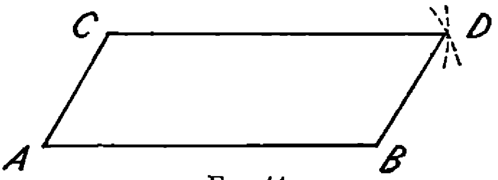


FIG. 44

7. Solve the equations

$$4x+7=6x-9\ ; \qquad 3(x-4)+2=6(x+2)-9\ .$$

PART III. RECTANGLE, SQUARE, TRAPEZOID, AND RHOMBUS

A. Study *Senior Mathematics,* II, 77–79, and be able to give correct meanings of the terms "rectangle," "square," "trapezoid," "isosceles trapezoid," and "rhombus."

B. Prove each of the following exercises and write the proofs into your notebook. Practice until you are able to demonstrate the proofs without referring to the textbook: Exs. 1, 2, p. 77, and Ex. 4, at the top of p. 78; Exs. 1–3, p. 78; all exercises on p. 79; Exs. 5, 6, p. 80; Exs. 16, 17, p. 83.

C. If you finish the work up to and including p. 79 ahead of the class, study pp. 80–83. The best students should solve Ex. 4, p. 78, and Exs. 10–15 on pp. 82–83.
D. Make a classification of the quadrilaterals studied in this unit.
E. Review and summarize what you have learned in Parts I, II, and III.
F. Test your understanding of Parts III with the following items:

1. Is a rectangle a parallelogram? *Answer:* A rectangle {is / is not} a parallelogram because ....
   ....
2. Is a square a rectangle? *Answer:* A square {is / is not} a rectangle because ....
   ....
3. Is a parallelogram a trapezoid? *Answer:* A parallelogram {is / is not} a trapezoid because ....
   ....
4. Put a check mark under the word, or words, for which the following statements are true:

	Quadrilateral	Parallelogram	Rectangle	Square	Trapezoid	Isosceles Trapezoid	Rhombus
Diagonals are equal............							
Diagonals bisect each other......							
Diagonals are perpendicular to each other..................							
Opposite sides are equal........							
Opposite angles are equal.......							
All sides are equal..............							
Only two sides are equal........							

5. To prove the diagonals of rectangle $ABCD$ (Fig. 45) equal, I must
   draw .... and ....
   and prove $\triangle$............ $\cong$ $\triangle$.............

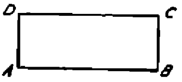


FIG. 45

6. In the square $MNPQ$ (Fig. 46) I may prove $\triangle MQP \cong \triangle MNP$ by showing that
   .... = ...
   .... = ...
   $\angle$............ = $\angle$............
   The triangles are congruent because ....
   ....

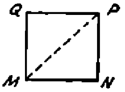


FIG. 46

7. A short way of proving problem 6 is..

8. Give the reasons called for below. In the trapezoid $MNPQ$ (Fig. 47), let $QM=PN$, and draw $QR \parallel PN$.

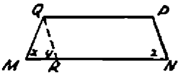


FIG. 47

Then	Reasons
$QM=PN$	(1)..
$PN=QR$	(2)..
$QM=QR$	(3)..
$x=y$	(4)..
$y=z$	(5)..
$x=z$	(6)..

The pupil's organization of the unit.—The pupil should be taught to keep up a daily review of the unit as far as he has studied it. He will then reach the end of the unit with a clear organization in his mind. For this reason pupils should be called upon occasionally during the study of the unit to give brief, well-organized oral reports on "what they have learned so far." To be prepared to give such a report at any time should be a standing assignment. This does away with the customary cramming at the end of the unit whose only purpose is to be able to pass the final test. The pupil should be told that when the end of the unit is reached he will be expected to write a complete organization without the use of the textbook and without further special preparation.

The following organization was written by a pupil and accepted as satisfactory by the teacher.

QUADRILATERALS

I. New terms
   A. In this unit I have learned the meanings of the following mathematical terms: "quadrilateral," "diagonal," "parallelogram," "rectangle," "square," "trapezoid," and "rhombus."
   B. The quadrilaterals are related to each other, as will be seen from Fig. 48.

II. Uses of the theorems taught in this unit

   There are many practical uses for the theorems taught in this unit. I have observed such uses
   1. In practical devices: the dentist's table and the ironing board.
   2. In engineering: bridges and derricks.
   3. In building: homes and factories.
   4. In designing: floor plans and borders for ornamentation.

III. Theorems taught in the unit

I have learned to give proofs of the following theorems:

A. Properties of parallelograms

1. The diagonal of a parallelogram divides it into two congruent triangles.
2. The opposite sides of a parallelogram are equal.
3. The opposite angles of a parallelogram are equal.
4. The consecutive angles of a parallelogram are supplementary.
5. The diagonals of a parallelogram bisect each other.

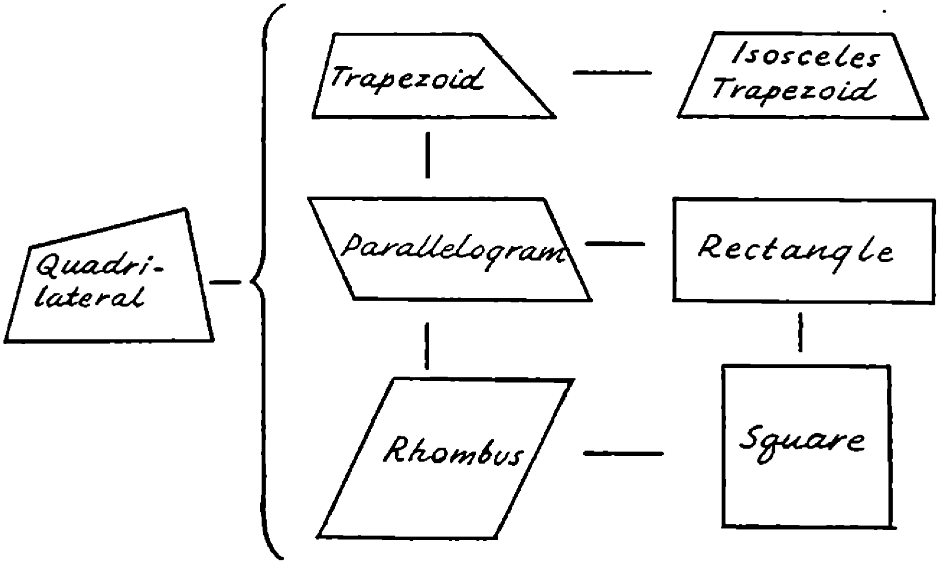


FIG. 48

B. Conditions which make a quadrilateral a parallelogram

A quadrilateral is a parallelogram

1. If the opposite sides are equal.
2. If two sides are equal and parallel.
3. If the diagonals bisect each other.
4. If the opposite angles are equal.

C. Properties of rectangle, square trapezoid, and rhombus

1. The rectangle
   a) The diagonals are equal.
2. The square
   a) The diagonals are equal.
   b) The diagonals bisect the angles of the square.
   c) The diagonals are perpendicular to each other.

3. The isosceles trapezoid
    a) The base angles are equal.
    b) The diagonals are equal.
4. The rhombus
    a) The diagonals are perpendicular to each other.
    b) The diagonals bisect the angles of the rhombus.

IV. Methods of proof

The following methods of proof have been used:

1. The method of congruent triangles.
2. The method by analysis.
3. The algebraic method of elimination.

V. Algebraic processes

I have reviewed the solution of linear equations in one and in two unknowns.

VI. Geometric constructions

I have learned to make the following constructions:

1. To construct a parallelogram.
2. To construct a rhombus.

The final test on the unit.—When evidence has been obtained that the class is ready, the final test should be administered. The test is devised to help the teacher secure further objective evidence with regard to the results of teaching. When the pupil writes an unsatisfactory test paper, he has not thoroughly assimilated the essentials of the course. The test shows definitely what the pupil needs to study. He receives further instruction and does additional studying. He is then tested again. Indeed, the process of testing and teaching should go on until the pupil has given evidence that he has acquired the essentials of the unit.

In administering the final test, several points should be kept in mind:

1. No time limits should be given for the test, and each pupil should be allowed ample time to finish.
2. Classroom conditions should be kept as nearly normal as possible.
3. The pupils' desks should be cleared of unnecessary materials.
4. Each pupil should be equipped with ruler, compasses, protractor, pencil, and eraser.
5. When a pupil has finished he should lay his paper aside. Later the teacher should take it up quietly without disturbing the others. An assignment of work for study should be made at the beginning of the period for those pupils who finish the test early.

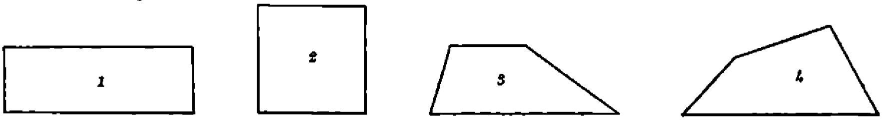
The final test for the unit on quadrilaterals follows.

FINAL TEST: QUADRILATERALS[5]

Name..Teacher..

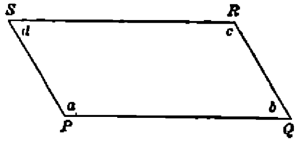
....

A. On each blank space below write the name of the polygon marked with the same number:



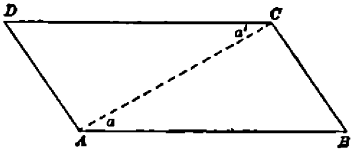
1. .... 3. ....
2. .... 4. ....

B. Complete the following statements and express each in symbols:
In the parallelogram *PQRS*
1. Two opposite sides are..........and..........
   In symbols:..
2. Two opposite angles are..........................
   In symbols:..
3. Two consecutive angles are.......................
   In symbols:..



C. The statements below are sufficient to prove that *if two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.* Arrange them in logical order and write the numbers of the ordered statements in the blank spaces:

1. $\triangle ABC \cong \triangle ADC$	(—) comes first
2. $AB \doteq DC$	( ) second
3. *ABCD* is a parallelogram	( ) third
4. $AC \equiv AC$	( ) fourth
5. Draw *AC*	( ) fifth
6. $a = a'$	( ) sixth
7. $AB \parallel DC$	( ) seventh
8. $AD = BC$	( ) eighth



[5] E. R. Breslich: *Mathematical Achievement Tests, Senior Mathematics, Grade X* (Chicago: University of Chicago Press).

D. For the following theorem draw the figure and state what is given and what is to be proved: *If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.*

Figure

Given:..

....

To prove:..

E. In the proof of the theorem that *the diagonal divides a parallelogram into two congruent triangles* the following statements and authorities are used. Draw a figure, and fill in the blank spaces the numbers of the statements and authorities arranged to make a complete demonstration.

1. Diagonal AC	4. $y=y'$	7. $ABCD$ is a parallelogram
2. $x=x'$	5. $AB \parallel DC$	8. $AC \equiv AC$
3. $\triangle ADC \cong ABC$	6. $AD \parallel BC$	9. Given

10. Two triangles are congruent if two angles and a side of one are equal to the corresponding parts of the other.
11. Any quantity is equal to itself.
12. Opposite sides of a parallelogram are parallel.
13. Alternate interior angles formed by two parallels and a transversal are equal.

Given:.. To prove:..

	Statements		Authorities
Proof *a*)	....	*a*)	....
b)	....	*b*)	....
c)	....	*c*)	....
d)	....	*d*)	....
e)	....	*e*)	....
f)	....	*f*)	....
g)	....	*g*)	....
h)	....	*h*)	....

F. On line EF construct a parallelogram having given the two adjacent sides equal to AB and CD:

A............B

C....................D

E....................................F

G. Solve for x:

1. $4x+8=19-6x$ 2. $4a+7=2+ax$
3. Eliminate a from the equations: $2a+3b=4$, $6a=5b$.

Directions for scoring the test.—To insure objective and uniform scoring a key is made (Table XLII).

TABLE XLII

KEY FOR SCORING

Test Items	Highest Possible Scores
A. Rectangle, square, trapezoid, quadrilateral	4
B. 1. Equal and parallel; $PQ \parallel SR$, $PQ = SR$, $PS \parallel QR$, $PS = QR$ 2. Equal, $P = R$, $S = Q$ 3. Supplementary, $P + Q = 180$, $Q + R = 180$, $R + S = 180$, $S + P = 180$	3
C. [5, 2, 7, 6, 4, 1] [8, 3]	2
D. *Given:* quadrilateral $ABCD$, $AE; EC$, $DE = EB$ *To prove:* $ABCD$ a parallelogram	3
E. *Given:* 7, 1 To prove: 3 Proof: [7, 9 / 5, 12 / 2, 13] [6, 12 / 4, 13] [8, 11 / 3, 10]	5
F. Correct drawing Accuracy	2
G. 1. $1\frac{1}{10}$ 2. $\frac{4a+5}{a}$ 3. $4\frac{2}{3}b = 4$	3

For the purpose of diagnosis of the test, the results are tabulated on the form shown in Table XLIII. As in the inventory test (Table XLI), the problems or test items missed are to be marked with a cross (×). The other spaces are to be left blank.

Examination of the horizontal lines and totals aids in diagnosis of individual difficulties. The vertical lines will show where the class as a whole needs further instruction. If every test item is checked, it is possible to tell not only which problems are difficult, but exactly where the pupil's difficulty lies.

Interpretation of the test results.—Table XLIV gives the results on the main divisions of the test for one of the classes that has taken it. It is evident from the totals of the vertical columns that the responses to C, D, and G were poor and that further class teaching needed to be done. On the other hand, the class responded well to Parts A, B, E, and F, and only one pupil needed further teaching on item A, one on B, one on E, and none on F.

TABLE XLIII

Diagnostic Record Sheet

Mathematics.................. Test.................. Unit..................

Period.................. Teacher.................. Date..................

Test Problems / Class Enrolment	A				B			C		D			E					F		G			Total Right	Satisfactory
	1	2	3	4	1	2	3	1	2	1	2	3	1	2	3	4	5	1	2	1	2	3		
1																								
2																								
3																								
4																								
5																								
6																								
7																								
25																								
26																								
27																								
28																								
29																								
30																								
Total																								

The class was retaught on the principles to be tested by C, D, and G.

The last column shows that the papers of seven pupils were considered unsatisfactory by the teacher. The pupils were called in for conferences and were given further instruction to correct individual deficiencies. In the case of Pupil 10 a second conference was needed before he could write a satisfactory paper.

It is not sufficient to know that a class or individual pupils have done good work or poor work in a test. The cause of failure must be definitely determined. This information may be derived from Table XLV which gives for one of the classes the results for the various

items of each division of the test and enables the teacher to locate definitely the places where the pupils had difficulty.

An examination of the horizontal lines aids in diagnosis of individual difficulties. The vertical lines show the items on which the class as a whole needs further instruction, as on C_1, C_2, D_2, and

TABLE XLIV

Mathematics: II Test: III Unit: Quadrilaterals

Period: 9:00 Teacher .... Date: December 5, 1932

Class Enrolment \ Test Parts	A 1	B 2	C 3	D 4	E 5	F 6	G 7	Total Right	Satisfactory
1				×				6	S
2							×	6	S
3			×				×	5	US
4								7	S
5				×			×	5	S
6								7	S
7							×	6	S
8							×	6	S
9			×	×			×	4	US
10			×	×			×	4	UUS
11								7	S
12		×					×	5	S
13			×	×	×		×	3	US
14							×	6	S
15							×	6	S
16			×				×	5	US
17								7	S
18							×	6	S
19			×					6	S
20								7	S
21			×				×	5	S
22							×	6	S
23			×	×			×	4	US
24								7	S
25								7	S
26	×		×	×			×	3	US
Total right	25	25	17	19	25	26	9	146	

G_2. If every test item is checked, it is possible to tell not only which problems are difficult but exactly where in each problem the pupil's difficulty lies.

The recitation.—The last step in the plan for the study of the unit is the recitation on the unit. The following procedure may be used. Topics are written on separate cards and distributed at the beginning of the class period. The pupils are then given five min-

TABLE XLV

Mathematics: II Test: III Unit: Quadrilaterals

Period........................ Teacher.. Date....................

Class Enrolment \ Test Items	A				B			C		D			E					F		G			Total Right
	1	2	3	4	1	2	3	1	2	1	2	3	1	2	3	4	5	1	2	1	2	3	
1				×							×	×									×	×	19
2													×	×							×	×	18
3				×				×	×											×			16
4																							22
5											×	×									×		19
6																							22
7																					×		21
8																					×		21
9				×				×	×		×										×	×	16
10								×	×		×	×								×	×	×	15
11																							22
12						×	×														×		19
13								×	×		×				×						×		17
14																					×		21
15																					×		21
16								×	×											×	×		18
17																							22
18																					×		21
19								×	×		×												19
20													×										21
21								×	×											×	×		18
22				×				×	×		×			×						×	×	×	17
23											×	×								×	×		15
24																							22
25																							22
26			×	×				×	×		×	×									×	×	14
Total right	26	26	25	21	26	25	25	17	17	26	17	21	24	24	25	26	26	26	26	20	9	20	498

utes to prepare recitations on the topics assigned to them. The assignments for the unit on quadrilaterals may be:

1. Discuss the importance and uses of quadrilaterals.
2. Tell how the quadrilaterals studied in the unit are related to each other.
3. Prove the theorem: A diagonal divides a parallelogram into congruent triangles.

4–17. An assignment of proofs of other theorems, as in assignment 3.

18. Explain the meaning and use of the congruent-triangle method in solving originals. [*Illustrate.*]
19. Explain how the analytic method of proof is used [*Illustrate.*]
20. Solve the system $\begin{cases} 5x+2y=4 \\ 3x-5y=21. \end{cases}$

21. Solve the equation $3x+2=5(x-5)-3$.
22. Construct a parallelogram having given an angle and the two including sides.
23. Construct a rhombus having given one angle and one side.

While the pupils prepare to recite, all textbooks should be closed. They may be permitted to take notes which they may use during the talks.

At the end of each talk the pupils and the teacher may ask questions to be answered by the speaker. The teacher should then make comments summarizing what has been said.

If necessary, more than one class period may be allowed for the recitation on the unit.

Examining the written work of the pupils.—Throughout the study of the unit, especially during the supervised study periods, the teacher has ample opportunity to observe daily the written work done by each pupil. Since the recitation marks the end of the unit, all the written work may be handed in for inspection. The teacher is thus given the opportunity to view each pupil's written work on the unit as a whole and to form a clear idea as to the character of the work done.

Summary.—The chapter has illustrated concretely the steps that may be taken in the study of a unit. The teacher or supervisor should have no difficulty in planning similarly the work of other units by using such steps as they consider helpful and worth while and by adding others that have not been illustrated in the foregoing description.

BIBLIOGRAPHY

Billett, Roy O. "Plans Characterized by the Unit Assignment," *School Review*, XL (November, 1932), 653–68.

Miller, Harry Lloyd. *Directing Study: Educating for Mastery through Creative Thinking.* New York: Charles Scribner's Sons, 1922.

Morrison, Henry C. *The Practice of Teaching in the Secondary School*, chap. iii. Chicago: University of Chicago Press, 1931.

Ruediger, William C. "The Learning Unit," *School Review*, XL (March, 1932), 176–81.

Stone, C. A., and Georges, J. S. "The Learning Products of a Unit of Instruction," *School Science and Mathematics*, XXXI (January, 1931), 58–69.

Sumner, S. Clayton. *Supervised Study in Mathematics and Science,* Part I. New York: Macmillan Co., 1922.

Sykes, Mabel. "Some Pedagogical Aspects of Geometry Teaching," *Mathematics Teacher*, XX (December, 1927), 466–72.

Waples, Douglas, and Stone, C. A. *The Teaching Unit.* New York: D. Appleton & Co., 1929.

CHAPTER XI

ARTICULATION OF JUNIOR AND SENIOR HIGH SCHOOL MATHEMATICS

The gap between elementary and high-school mathematics.—It has been shown in chapter iv that algebra, geometry, and trigonometry as taught in our high schools were originally organized for students in colleges. Later they were taken over by the college preparatory schools and high schools, being offered at first in the Senior and Junior years. Gradually they were moved downward to the early years of the secondary schools. They were presented to high-school students almost as they had been taught in the colleges. When it was found that high-school pupils encountered serious difficulties in the study of algebra and geometry, efforts were made to reorganize or to reconstruct these courses.

Because our school system was organized on the 8–4 plan, it seemed to be practically impossible to find a satisfactory solution of the problem. For on account of the gap existing between the elementary and high schools the teaching of algebra and geometry had to be deferred to the ninth and tenth school years. This made it necessary to crowd a large amount of algebra into a year's time and to force the pupil to advance at a rate too rapid for understanding and assimilation. Since all of the time of the second year was needed to cover the large content of demonstrative geometry, algebra had to be completely dropped during that time.

The junior high school movement simplified the problem of reorganization.—Attempts have been made by individual schools to solve the problem by offering some algebra in the last year of the elementary school. However, such efforts usually failed for two reasons: high-school algebra was not easily adapted to the interests and abilities of elementary-school pupils; and high-school teachers were unwilling to accept algebra taught in the eighth grade as an equivalent of high-school algebra. Often it was completely disregarded and the work done in the eighth grade was repeated in the ninth.

The junior high school offered a real opportunity for the reorgani-

zation of secondary-school mathematics. It made it possible to extend the period of secondary education downward, to give instruction in algebra and geometry at an early period, and to distribute the content of these courses over several years. This involved a reconstruction of the mathematical curriculum of the secondary school, which offered introductory work in algebra and geometry in Grades VII and VIII and the more advanced work in the later grades.

The new gap formed between junior and senior high school mathematics.—The reorganization necessitated the training of teachers qualified to teach the new courses. Elementary-school teachers were unfamiliar with the mathematics of the secondary school, and high-school teachers were not accustomed to pupils of elementary-school age. To satisfy this need courses in junior high school mathematics were offered by the universities. A new type of mathematics was being developed for the junior high school, while the senior high school continued to offer the traditional courses in plane geometry, advanced algebra, solid geometry, and trigonometry.

The administration and supervision of the mathematics in the junior and senior high schools was usually not carried on by the same supervisor and the teaching was done by two distinct faculties. Neither division knew much about the other.

The development of two types of secondary-school mathematics, the separation of the junior high school from the senior high school, and the creation of two distinct faculties teaching the courses tends to form a gap in secondary-school mathematics fully as serious as that which formerly existed between the mathematics of the elementary and high schools. Steps should be taken to bring about a closer co-operation.

Uniformity of administration is an essential factor in securing improvement.—An ideal situation is to have but one department and one supervisor to direct it. When there are two supervisors they should feel compelled by professional interest to develop a program of unifying and articulating the work of the two departments. Mutual understanding and harmonious co-operation of the teachers will not be difficult to secure if there is uniformity of administration. To attain it will require frequent joint departmental conferences in which the problems of concern to both divisions are freely

discussed. The junior high school teacher will thus become acquainted with the content and objectives of the higher courses and will be able to do his work intelligently. The senior high school teacher will acquire an understanding of the problems and limitations of pupils of the lower grades and will become willing to assume a just share of responsibility toward them. Both will grow in knowledge of subject matter. They will reach agreement as to methods of teaching so that the pupil passing from one institution to the next higher experiences no sudden and radical changes in teaching and study.

Continuity of instruction is imperative.—Continuity in secondary-school mathematics is exceedingly important. Teachers should therefore cease to regard junior and senior mathematics as distinct types. The two should be thoroughly merged. One course of study should be planned to include both, and the details should be worked out by joint committees selected from both faculties. As the pupil passes from the junior to the senior high school, the change in mathematics should be no greater than that usually experienced when he advances a grade. The senior high school should begin exactly where the junior high school course ends. The pupil's attitude toward senior high school mathematics will be friendly, for he has had enough mathematics to know what it is like and since he has deliberately chosen to continue the study it may be assumed that he is interested in it, feels capable to do the work, and believes that he will be benefited.

Articulation of junior and senior high school geometry.—High-school teachers of mathematics are sometimes heard to complain that it is difficult to teach geometry to pupils coming from the junior high school. They assert that the most interesting part of geometry has been taught in the lower school and that the attitude of the pupil is one of indifference and exceedingly hard to change. Usually the difficulty arises from lack of co-operation of the two departments, and the solution lies in establishing better understanding between them.

It is generally agreed that the course in demonstrative geometry is too crowded for attaining the major objectives with the maximum number of pupils. Many pupils capable of doing high-school work never develop the ability to solve original exercises and have to be

satisfied with learning and repeating the finished proofs found in the textbooks. It would seem that teachers of high-school geometry would welcome the opportunity to have some of the geometric work done in the lower grades.

From the point of view of the junior high school the transfer is not only desirable but imperative, because pupils should be taught to understand and interpret the geometric situations which they meet in the activities of everyday life and in various school subjects. Familiarity with the most common geometric figures, knowledge of an essential geometric vocabulary, appreciation of the beauty of geometric forms in nature, training in space imagination, ability to measure and to estimate size, acquaintance with a number of important geometric facts and relationships, and skill in the use of the geometric measuring instruments are all included among the important social and vocational objectives of the junior high school. Not only is this type of geometry needed in the activities of pupils, but it is also an excellent preparation for, and introduction to, high-school geometry. Pedagogically it is best to teach it during the junior high school period.

The teaching of geometry is not always taken as seriously by the junior high school teachers as the more traditional work of arithmetic. It is not a matter that may be neglected. The content should be so carefully selected and organized and so thoroughly taught that the senior high school teacher may know exactly what has been taught and what knowledge and information he may take for granted in his courses. It should be subjected to a careful program of testing and reteaching to secure objective evidence of what has been learned. When this is done the study of geometry may proceed without interruption as the pupil passes from the junior high school to the senior high school. Several examples will be given to illustrate the relationship between junior and senior high school geometry and the need for articulating the two.

The fundamental concepts of geometry.—Lack of understanding of the meaning of geometric concepts is the source of much difficulty in high-school geometry. A definition of the new term and a few illustrations are often all that is given. Furthermore, a considerable number of unfamiliar terms are being crowded into the beginning of the course. To the pupil this type of work is uninteresting and

often distasteful. Failing to catch the meanings of the new concepts, he memorizes statements which he does not understand. Naturally he is unable to make correct use of them when they occur later in the logical discussions. He becomes discouraged and fails.

When the meanings of concepts are acquired in the junior high school, this may all be changed. Here the new terms arise in concrete situations and activities. The learner is not hurried. There is ample time to observe, to compare, to measure, to draw, and to analyze. Understanding and satisfaction are the results. A complete list of the concepts whose meanings have been established in the junior high school should be available to the high-school teacher for inspection. Beginning with this body of information, he will add other terms gradually as need for them arises in his own course.

Constructional geometry.—Traditionally the geometric constructions form an essential part of high-school geometry. The present tendency is to introduce them early in the course rather than to defer them until they may be proved by logical demonstration. It is therefore a simple matter to take the next step and to move at least the so-called fundamental constructions downward into the junior high school. They give the pupil excellent practice with the geometric instruments. He will use them in making diagrams and designs. They will be used with sufficient frequency to be learned and retained. It will not be necessary to repeat this type of work in high-school geometry. They will be reviewed when they are employed in constructions of a more complicated type.

Simple geometric facts.—Through his daily life-experiences the pupil has become acquainted with numerous geometric facts and theorems before he enters the high school. Others he has learned in mensurational geometry in the elementary school. As far as he knows, the truth of these facts has never been questioned. He has made much use of them in the solution of problems. At the beginning of demonstrative geometry he is practically told that he should forget all the geometry he has previously acquired and that he will not be permitted to use any fact that has not been established by logical proof. This seems to him most illogical and confusing. Often he becomes openly antagonistic. He is in no frame of mind to appreciate or enjoy the beauty of the logical system of geometry.

The difficulty is easily removed by a little co-operation between the two schools. An agreement should be reached as to the particular facts to be taught in the junior high school. These facts should then be included among the assumptions with which the course in demonstrative geometry begins. This increases the number of assumptions. On the other hand, it enables the teacher to organize a course which observes a truly logical sequence. It may call for slight readjustments of the theorems in the textbook. Some of them will be reduced from the ranks of basic theorems and included among the exercises. No harm will be done, as this conforms to the tendency of recent years to diminish the number of basic theorems.

Developing the meaning of a logical demonstration.—To a large extent demonstrative geometry is really a course in logic. The pupil entering the tenth grade has previously developed reasoning ability. He can hold his own in an argument and in a debate. However, the formal demonstration so prominent in tenth-grade geometry is new to him. He comes upon it abruptly without anything just like it in his former experiences. Many pupils find the step from junior high school geometry to logical geometry very difficult and go about their work aimlessly for weeks before they grasp the idea. Often the inability to take this step is the cause of failure.

The problem is serious enough to call for the careful consideration of the teachers of both schools. Somewhere in the course provision should be made for a period of instruction in which space and logic are being gradually joined. Modern textbooks on geometry take recognition of this need by offering an introductory chapter preceding the formal geometry. However, the results are not at all satisfactory because to develop the meaning of logical demonstration takes more time than may be devoted to it in an introductory chapter.

The chances for solving the problem in the junior high school are much better. There the progress in the study of geometry is slow. Ever so many situations present themselves in which facts may be easily established by simple reasoning cycles. This work can be made attractive to the pupils. Gradually they will see and appreciate the power and advantage to be derived from the logical proof. It is possible to work out a detailed, definite program which

will lead the pupil gradually from the method of direct observation through a period of informal reasoning to the stage of demonstrative geometry. The last belongs properly to the senior high school. The first two should be provided in the junior high school.

Continuity of instruction in arithmetic during the secondary-school period.—The fundamental processes of arithmetic are taught in the first six grades of the elementary school. However, arithmetical maturity is usually not attained by pupils before the junior or senior years of the high school. It is not unusual to find high-school Seniors who have not advanced beyond the level of the fifth-grade pupil in arithmetic. On the other hand, pupils are found in the seventh grade who have already attained maturity in arithmetic. It is evident that instruction in arithmetic must continue for many pupils when they are taking secondary-school mathematics even though a formal course should not be offered. Anyone who has examined the results of arithmetic tests administered to high-school pupils will agree to this. High-school teachers cannot wave aside the responsibility. Arithmetical deficiencies of individuals and of classes should be discovered by means of tests given each year to each grade. Steps should then be taken for remedial work. Every teacher from the seventh grade up should know definitely his obligations as to arithmetic, and the faculties of both schools should co-operate to the fullest extent in the program of bringing about arithmetical improvement and growth.

The foundation of senior high school mathematics is laid in the junior high school.—It is not intended to give the impression that the major aim of junior high school mathematics is preparation for the later courses. The content of the curriculum at any stage should be determined first of all by the needs of pupils in their activities in and out of school. However, this does not imply that the needs of later life and of the more advanced mathematics should be disregarded. For frequently the immediate and deferred needs are identical. They demand and deserve careful consideration. They raise problems of teaching that are of importance to the pupil's successful continuation of the study. If the faculties of both schools are interested and co-operate in the solution of the problems, a correct start will be made and instruction will continue without interruption and with a minimum loss of time and effort.

It is evident that teachers are not qualified to give instruction in junior high school mathematics unless they possess a thorough knowledge of the mathematics taught in the senior high school and in the junior college. Without it they will lack the point of view, and their work will be ineffective. A few examples will be offered to illustrate the correctness of this assertion.

Continuity in the teaching of graphical representation.—The teaching of graphs in secondary-school mathematics is a relatively recent innovation. Originally they belonged to the field of higher mathematics, but they found the way into high-school mathematics and from there into the junior high school. Today they are taught, or used, in all mathematics courses and in some of the other school subjects. Their importance is recognized not only in school work but also in everyday life. The teaching of graphs should begin in the junior high school and should continue in the senior high school. The responsibility for teaching them is to be assumed by both departments, each taking its share. On the part of the teachers this presupposes a thorough knowledge of the subject.

In the junior high school the pupil should acquire a knowledge of different types of graphs used to represent numerical facts. He should attain considerable ability to employ them in a variety of situations and to understand and interpret relationships when they are pictured graphically. He should become familiar with the bar graph, line graph, and circular graph, and learn to solve linear and quadratic equations by graphical methods. This cannot be accomplished with an occasional chapter on graphs. They should be used constantly.

Graphical work begins when pupils draw line segments of given lengths and find lengths of given line segments by estimating and by applying the ruler. From this simple drawing and measuring of single line segments it is but a small step to the statistical graph. This relation is not always recognized by the teachers, and the introductory work is often slighted and even omitted because its importance is not realized. The place for teaching it is in the early part of the seventh grade.

The drawing and measuring of line segments should be assigned a prominent place in mathematics not only because it is an introduction to graphical work but on account of its usefulness for other

purposes. For example, it is essential to an understanding of such concepts as ratio of geometric magnitudes, proportion, similarity, and trigonometric function. If in trigonometry the sine function is introduced as a ratio of two sides of a right triangle, it is assumed that the pupil knows what "ratio of two sides" means. This seems justifiable because the term is used over and over in tenth-grade geometry. The chances are that the teacher of geometry also presupposed an understanding which did not exist and failed to explain the term. Most likely he disposed of it by saying that the ratio of two line segments is the ratio of the numerical measures. However, the explanation means little or nothing to the pupil who has not had actual experiences with measuring and of finding ratios by dividing one measure by another. Thus, the drawing and measuring of line segments is an introduction not only to graphical representation but to geometry, trigonometry, and analytics.

A program for teaching graphs during the entire period of secondary-school mathematics should be worked out to make them a powerful tool in the study of mathematics and a mode of thinking which makes abstract relationships concrete. Not very long ago teachers objected to graphical representation on the ground that it introduced rather than removed difficulties. In textbooks on algebra the graphical solutions of equations were cautiously placed in a chapter by themselves where they could easily be omitted. In trigonometry the graphs of the trigonometric functions were regarded as frills. They were taken up when the course was finished, if time permitted.

The tendency today is to make graphs useful and to teach them not as curiosities but as an essential part of the course. The trigonometric functions illustrate this strikingly. To give the learner a clear conception of the functions and their relationships to each other, teachers make use of a variety of devices which may all be replaced by a simple graph. The student who understands the sine graph sees at a glance that the sine function is positive in the first two quadrants and negative in the last two; that it assumes all values from 0 to 1 as the angle changes from 0° to 90°; that the maximum and minimum values are $+1$ and -1; that the sine of $(n\times 90\pm x)$ is numerically equal to sine x or cos x according as n is even or odd; and that it is a periodic function. Moreover, the graph

aids the mind to retain all these facts easily and, if in doubt, the student may reproduce them in a moment by making a rough sketch of the graph.

The importance of graphical representation cannot be overlooked, and the co-operation of senior and junior high school teachers is needed if it is to be taught effectively.

The teaching of the formula calls for united effort of junior and senior high school teachers.—The formula deserves a prominent place in all mathematics courses. Instruction in it should start early and never cease, for it takes time to develop the various abilities with formulas.

One of these is the ability to derive formulas. In senior high school mathematics formulas are rarely used unless they have been derived by the pupils. The same is true for the formulas which occur in the science courses. Experience shows that the deriving of the formula aids the pupil in understanding it and gives him confidence in its use. For this reason the pupils generally develop the progression formulas, the quadratic-equation formula, the binomial theorem, and the laws of uniform motion and of falling bodies.

The first formulas with which pupils become acquainted in the study of mathematics are simple and are easily derived. This is true particularly of the formulas for finding areas of the common plane figures, volumes of solids, percentages, and interest. Nevertheless, teachers and textbooks often neglect to derive them. They merely state the formulas, explain the meaning of the symbols, and proceed to the applications. Thus the pupil's conception of the formula is necessarily limited since he sees in it little more than a rule or device which he must learn to manipulate to find the required answers to given problems. The opportunity for training in deriving formulas has been neglected.

The evaluation of formulas is another process to be stressed throughout the entire period of secondary-school mathematics. It deepens the pupil's understanding of the mathematical symbols and relationships contained in formulas and gives practice in substitution and in the fundamental operations. This phase of formula work has always received its share of attention from the teachers.

The transformation of formulas is a third ability to be developed. Training may begin as soon as the pupil has derived the first

formulas. Thus he should be taught early to solve $c=2\pi r$ for r; $i=prt$ for p, r, or t; $d=rt$ for r or t; $A=bh$ for b or h. It will take a large variety of formulas and problems, constant practice, and teaching extending over a period of years to develop proficiency in transforming formulas.

A fourth phase of the study of formulas is the development of power to understand relationships contained in them. This is essential to mastery of the concept. Thus the pupil should see that the value of c is doubled in $c=2\pi r$ if r is doubled, but that A in the formula $A=\pi r^2$ becomes four times as great if r is doubled; that t in $t=d/r$ is increased if r is decreased, but that it is decreased if r is increased.

Relationships in formulas have received little attention in teaching, with a few exceptions as in the case of the quadratic formula. Here it is usually shown that the character of the roots of an equation is related to, and depends on the values of, the coefficients. Although the relationship is simple, pupils generally experience considerable difficulty in comprehending it. The reason is that they have not been trained to examine formulas as to the relationships existing among the variables. The preceding courses should provide a considerable amount of such training.

At all levels mathematical instruction should aim to develop power to solve verbal problems.—Proficiency in solving problems is regarded by many teachers as the most important function of the teaching of mathematics. They would make the problem the outstanding feature of each of the various courses. They would introduce new processes with problems in which they occur and justify them on the ground that they are needed in solving the problems. Others emphasize the verbal problems on account of their informational value. They regard them as the most useful and most interesting part of mathematics.

On the other hand, the teaching of verbal problems has been severely criticized for two reasons: the failure to select more problem material that will be understood and appreciated by the pupils and the lack of an effective technique of teaching pupils to solve problems. Both criticisms necessitate the careful attention and cooperation of the teachers. Training in problem-solving should continue without interruption in all mathematics courses. Problems

should be more than mathematical puzzles. They should give the pupil a feeling of reality and impress him with the value and importance of mathematics.

Equally important is an effective technique of training pupils to solve problems. The literature relating to difficulties experienced by pupils in solving verbal problems is full of helpful suggestions. In the beginning of the secondary-school period problems are solved by arithmetic. Later the algebraic method is used almost exclusively. Somewhere within this period the transition must be made from the first method to the second.

However, the two methods have certain steps in common, and practice in them should be provided throughout the entire period. This includes training in reading verbal problems understandingly; in comprehending the social situations contained in them; in analyzing the content of problems as to the facts that are stated or implied; and in separating the known facts from those that are to be found. In addition to this the arithmetical method requires training in selecting the right process and the algebraic method calls for practice in recognizing relationships and in deriving the equation, or equations, for solving the problems. The change from the first to the second method should be gradual. It is unwise to force upon the pupil a method which he does not appreciate. If he is led to see the superiority of the algebraic over the arithmetical method, he will prefer to use it.

The first step toward the algebraic method will be to use literal numbers to denote the unknown or required numbers. Somewhat later all unknown numbers will be expressed in terms of the same literal number to simplify the verbal statements. As a further simplification the equation should be introduced. For the first problems the equations will be simple enough to be solved by inspection without the use of axioms or laws. As they increase in complexity the axioms will be employed in solving. Progress should be gradual, and sufficient time for assimilation should be allowed each step. At the end of the junior high school period the pupil should be quite proficient in solving verbal problems by algebraic methods.

Functional thinking is to be stressed throughout the period of secondary mathematics.—In the past much more attention has been given to the computational and formal phases of mathematics than

to the informational, cultural, and functional aspects. Without disregarding the importance of the first the tendency in modern teaching is to give more recognition to the second. Thus the student of mathematics should acquire power to think independently, learn to enjoy the beauty of mathematics, and appreciate the subject as one of the great achievements of man. One of the objectives of the teaching of mathematics is the power to do functional thinking.

Until recently it was assumed that the function concept belongs to the field of higher mathematics. It is now generally agreed that training in functional thinking deserves a place in secondary-school mathematics. Indeed, anyone who watches the activities of young children will find evidence of functional thinking in the earliest stages of school work and in the everyday experiences of the pupils. These experiences should be utilized in elementary-school mathematics to give training in functional thinking. In the junior high school opportunities for further training are numerous, especially in the study of algebraic formulas and the geometric theorems which express relationships between variables. At the senior high school level the function concept may then be made the central theme of mathematics. "The teacher should have the idea constantly in mind and the pupils' advancement should be consciously directed along the lines which will present first one and then another of the ideas upon which finally the general concept of functionality depends."[1]

Summary.—Some of the major objectives of the teaching of mathematics have not been acquired and cannot be attained by pupils as long as the great ideas are taught as isolated topics or as separate chapters distributed at various places during the secondary-school period. Mastery of these ideas develops slowly. They must be presented repeatedly in various situations and at different levels. Each teacher must know what has preceded his course and build on it. Each must be familiar with that which is to follow and pave the way for it. The gap between junior and senior high school mathematics will thus be eliminated. The accomplishment of these results requires the fullest co-operation of the teachers of both departments.

[1] *Report of the National Committee on Mathematical Requirements* (1923), p. 12.

BIBLIOGRAPHY

Articulation Committee for Mathematics of the Lake Shore Division of the Illinois Teachers Association. "A Composite Course for Seventh and Eighth Grade Mathematics," *Mathematics Teacher*, XV (January, 1922), 43–48.

Newlin, Jesse H. "Articulation of the Junior and Senior High School," *Proceedings of the National Education Association* (1927), pp. 765–71.

Thiele, C. L. "Coordination between the Junior and Senior High Schools," *Second Yearbook of the National Council of Teachers of Mathematics* (New York City: Bureau of Publications, Columbia University, 1927), pp. 171–72.

Van Denberg, Joseph K. "Articulation of Junior and Senior High School Mathematics," *Mathematics Teacher*, XIV (February, 1921), 88–94.

CHAPTER XII

UNIFIED MATHEMATICS AND THE CHANGING CURRICULUM

The curriculum must be adapted to the changes in the social order.— The American high school developed out of the needs of society, and its rapid growth is accounted for to a large degree by the constant endeavor to serve the needs of society. The number of pupils has increased at a rate passing all expectation until now more than half of the population of high-school age is enrolled in the secondary school. Twenty-five years ago it was an unusual event when a member of the family graduated from the high school. Today the family expects all of the children to become high-school graduates and, if the income permits, to go to college. In the present depression when jobs are few and industry does not need them, ever so many high-school graduates are appealing to the school to help them change enforced idleness into an opportunity for further education and improvement. They are asking for readmission, and the disappointment is keen when they find that the schools are not able to accommodate them.

The high school has not been free from criticism, and only too often criticism was justifiable. At times the usefulness of a high-school education has been questioned. Nevertheless, the record shows that the school has served society well and that it has been prompted by a genuine desire to eliminate objectionable features which gave rise to just criticisms, and to increase its usefulness by making the curriculum flexible, and by adapting it to the changes in the social order. Among numerous adjustments should be mentioned the offering of alternative curriculums to satisfy the varying needs of pupils; the meeting of the demands for vocational courses and guidance; and the adoption of extra-curricular activities as a part of the regular school program. New courses have been introduced in the natural sciences, the social sciences, music, and art. If necessary, changes have been made in the organization of the school. Every department has examined its courses to discover and

discard weaknesses. Valid aims and purposes have been formulated and scientific experiments have been conducted to determine the most suitable instructional materials and the most effective methods of teaching. The teachers of the various subjects have recognized that no subject should be allowed to stand still when the entire system of school organization is undergoing reconstruction.

Reforms in mathematical instruction.—It is to be expected that mathematics, being one of the oldest and first subjects to be placed on the high-school curriculum, would receive its share of criticism, and that it should play an important part in the reconstruction of the curriculum throughout the period of readjustment. Progress has been made but at times it has seemed very slow. It was retarded by the extreme conservatism of many administrators and teachers; the lack of preparation of the teachers who were to put new ideas into practice; the difficulties encountered in getting the administration to accept the new courses; the administrative difficulties in transferring pupils from one school to another; the rigidity of college entrance requirements; the failure of college entrance examinations to keep up with modern tendencies, and the resulting fear of teachers that pupils taking the new work would be unable to make passing grades in such examinations.

Regardless of retarding influences, much has been accomplished. There has been no lack of leadership, and during the period of the last fifty years many of the recommendations of these leaders and of several important committees have been accepted. They have pointed the way to far-reaching reforms. Future history will probably give credit for the reforms in secondary-school mathematics to the famous Committee of Ten on Secondary School Studies, to John Perry, to Felix Klein, and to Eliakim Hastings Moore. Their recommendations were discussed widely and received enthusiastically. Unfortunately, so far not all of them have been put into practice. Much remains to be done. Today there is as much need as ever for strong leadership or a far-seeing national committee with the courage and ability to finish the work so well started thirty to forty years ago.

Report of the Committee of Ten on Secondary School Studies.[1]—The

[1] *Report of the Committee of Ten on Secondary School Studies* (National Education Association; New York: American Book Co., 1894).

subcommittee of the Committee of Ten in 1893 which formulated the *Report* on mathematics was composed of outstanding mathematicians of that period. The *Report* represents the best thought and suggestions which they were able to offer. It was ahead of the times and some of the recommendations are just beginning to be appreciated and accepted. The following are typical: systematic instruction in concrete and experimental geometry should begin at the age of ten and occupy one hour per week for at least three years; systematic instruction in algebraic symbols and in simple equations should be offered before the pupil reaches the ninth grade; and algebra and geometry in the plan should not be taught as separate subjects but in connection with arithmetic.

The Perry movement.—During the years 1900–1902 Professor John Perry of England delivered a number of lectures[2] in which he advocated among others: steady emphasis on the practical uses of mathematics; selected applications from other school subjects; the use of the experimental method in developing principles of mathematics; and the early teaching of the useful parts of the various mathematical subjects. Perry's views were strongly indorsed by the teachers of America. His influence has been powerful and far reaching in the teaching of secondary-school mathematics. He gave impetus to three movements: to connect mathematics closely with other school subjects and with materials of interest to boys and girls; to bring into the lower courses many of the simple but exceedingly valuable notions usually developed in the upper courses; and to make mathematics concrete by deriving abstract concepts and relations from concrete experiences, thereby making them understood so they may be mastered by the pupil. At the time Perry was advocating his views, similar suggestions were made by leading mathematicians in various countries of the civilized world.

Klein's recommendations.—Professor Felix Klein of Germany was well known to American teachers of mathematics. He stressed two principles of organization:[3] that algebra and geometry be joined

[2] "The Teaching of Mathematics," *Educational Review*, XXIII (1902), 158–81.

[3] "Ueber den mathematischen Unterricht der Hoeheren Schulen," *Jahresbericht der Deutschen Mathematiker Vereinigung* (1902), pp. 128–40.

by making the function concept the unifying idea in mathematics, and that a psychological arrangement of subject matter be insisted on.

Moore's principle of organization.—Professor E. H. Moore of Chicago felt that the solution of the problem of improving secondary-school mathematics should be sought in a reorganization that would "abolish watertight compartments by which arithmetic is taught in one, algebra in another, geometry in another, and trigonometry in still another." He advocated the teaching of mathematics as one subject, in which each division should help and illuminate the others.[4]

The foregoing recommendations have aroused much discussion and a great deal of constructive activity among the teachers of mathematics who undertook to put them into operation. Committees were appointed to study the merits of the various plans. Writers of textbooks have incorporated the new ideas into their courses. Interest has been preserved by numerous articles which have appeared in the yearbooks of the National Council of Teachers of Mathematics and in the mathematical journals.

Improvements in mathematics resulting from general educational movements.—Attention has been called so far only to reforms in the teaching of mathematics which have come from within, i.e., from the teachers of mathematics themselves. However, the subject has made many valuable contributions by participating in general educational movements of national scope. It has rendered valuable assistance to the teaching movement by producing mathematical achievement tests, mathematical ability tests, diagnostic tests, and prognostic tests. With the coming of the junior high school, serious attention has been given to the construction of new courses and textbooks to be used in the new institution. Techniques of supervised study and individual instruction to supplement group teaching have been worked out more successfully in the field of mathematics than in most of the others. When the high schools were called upon to state the objectives to be attained in teaching, some of the best studies were made in mathematics. The mathematical and educational literature discloses that the teachers of

[4] "On the Foundations of Mathematics," *Science*, XVII (March, 1903), 401–16.

mathematics have readily co-operated in all important educational movements and that the subject has profited by it.

Weaknesses in mathematical instruction.—Although leaders in mathematics have launched a number of movements to eliminate deficiencies in teaching and organization of the subject and have shown ways of introducing general educational reforms, the old criticisms have not ceased and dissatisfaction still persists. Apparently the reforms have not been as far reaching as necessary. They have been introduced successfully in certain schools while many teachers and schools have taken but little interest in them.

The typical complaints about the results of teaching mathematics are:

1. Protests against the large number of failures in the mathematics courses required for graduation.

2. Dissatisfaction with the small amount of mathematical knowledge and skills retained by those who have received passing grades.

3. Poor preparation of high-school graduates who study college mathematics.

4. Dissatisfaction of teachers of science in high schools and colleges because students are unable to use the mathematics which occur in these subjects.

Invariably it is pointed out by those who make the criticisms that a knowledge of the simplest facts and processes is all that is expected and that the deficiencies are found in the fundamental aspects of the subject. The four criticisms will be examined in detail.

Failures in mathematics.—The large percentages of failures in high-school algebra and geometry have been a constant source of dissatisfaction among pupils, parents, and school officials. For years much has been said and written to bring about a reduction of failures of pupils, but the percentages continue to be high. Fifteen to 30 per cent of pupils fail commonly in mathematics, and often the percentages run as high as 50 per cent.

Many teachers of mathematics refuse to be disturbed by these large percentages. They do not regard them as a serious matter. They point to the record of high percentages of failures in other departments and assert that the responsibility rests with the pupils

and with the administrators, who admit too many poorly prepared pupils to the schools. However, there is little comfort in this argument to those who believe in the values which pupils may and should derive from the study of mathematics. High percentages of failures in any subject operate against it in two ways: school administrators will advise pupils against taking courses in the subject and pupils will avoid them because of fear of failing. Hence it becomes increasingly difficult for a subject to meet the competition with other subjects in which pupils are able to profit sufficiently from instruction to make passing grades.

It has been charged by school officials that failures in mathematics, together with the withdrawals of pupils who are afraid that they will fail, contribute heavily to the elimination of pupils from the school who are thus deprived of a general education. As a solution of the problem they are recommending further reductions of the mathematical requirements for graduation. The requirement used to be two and one-half years of mathematics. At the present time two years is the maximum. Many schools require but one year and others have dropped all mathematical requirements. This is not always to the best interest of the pupil. Recently a dean who advises premedical students in a large university deplored the fact that so many are now entering college with only one unit of algebra and one of geometry. "They inform me," he said, "that their high-school principals and advisers have assured them that this much high-school mathematics is an adequate preparation for premedical work in college. The fact is that they need in addition at least a course in intermediate algebra and that trigonometry should be advised. Since the university does not offer courses in intermediate algebra, the students have to make it up either by correspondence or with a tutor." Deans advising students who wish to sign up for physics, chemistry, astronomy, and mathematics report the same difficulty.

The extent to which other subjects are competing successfully with mathematics is indicated in certain data on percentages of pupils taking high-school algebra and geometry, published recently in the Biennial Survey of Education.[5] In interpreting the data it

[5] *Biennial Survey of Education* (Washington, D.C.: U.S. Office of Education, 1930), Bull. 16, pp. 1057–58.

should be kept in mind that during the period from 1890 to the present time the enrolment of pupils in secondary schools has been increasing. In 1900 it was about a half-million, and now it is close to four millions. Percentages in algebra at first increased from 45.4, reaching the high peak of 57.5 in 1905. However, in the next ten years the relative gain made during the preceding fifteen years was practically wiped out. The percentages had not ceased to decrease in 1928 when it was 35.2. The situation in geometry is similar. The increase of percentages runs parallel to that of algebra, beginning with 21.3 in 1890 and reaching the peak of 30.9 in 1910. The gains made in the first twenty years were more than wiped out in the following fifteen years. In 1928 the percentage was 19.8.

TABLE XLVI

PERCENTAGES OF PUPILS TAKING HIGH-SCHOOL ALGEBRA AND GEOMETRY

	1890	1895	1900	1905	1910	1915	1922	1928
Algebra (including intermediate)...	45.40	54.27	56.29	57.51	56.85	48.84	40.15	35.22
Geometry........	21.33	25.34	27.39	28.16	30.87	26.55	22.68	19.80

Of course, owing to the large increase in total enrolment, the falling-off of percentages does not mean a correspondingly large decrease in the number of pupils taking algebra and geometry. It does mean that forces are operating which are unfavorable to mathematics. Not the least of them are the high percentages of failures; the growing competition with the natural and social sciences, literature, music, and art; and a change in the friendly attitude of the school officials. The first should receive the most serious attention of the teachers. It calls for improved methods of teaching and for a selection and organization of instructional materials which is adapted to the abilities and interests of the present-day generation of pupils.

Pupils do not retain the mathematics taught in the secondary school. —Teachers of college courses in science and mathematics have always stressed the importance of a thorough mathematical preparation. This fact has greatly strengthened the position of mathematics in the secondary-school curriculum. It is therefore disturbing to have expressions of criticism and dissatisfaction come from a

group which naturally should be the most friendly. Frequently complaints are made that mathematical difficulties contribute heavily to mortality in courses in the college sciences. In a recent conference several college instructors expressed the opinion that the study of first-year algebra "seems to be a total loss to many students."

When asked for specific illustrations of common mathematical deficiencies, many examples were given, among which the following are significant:

1. Arithmetical deficiencies
   a) Inability to divide one decimal fraction by another, as .00000432÷.00016.
   b) Inability to extract the square root of a number, and lack of understanding of the principles on which the process is based.
   c) Inability to determine which of two common fractions is the greater, as in $\frac{2}{4}\frac{3}{5}\frac{1}{1}$ and $\frac{2}{4}\frac{3}{5}\frac{2}{2}$.
   d) Lack of the reasoning power necessary to solve simple verbal problems.
2. Algebraic deficiencies
   a) Errors in adding or subtracting algebraic fractions, as $\frac{a}{n}+\frac{b}{n}$, $\frac{1}{x+1}-\frac{1}{x}+\frac{1}{x-1}$.
   b) Inability to expand correctly $(a \pm b)^2$ and $(a+b)(a-b)$.
   c) Inability to solve equations, as $2x=14$, $3[2+4(1-x)](6-1)=0$, $\frac{\frac{3}{4}}{\frac{5}{x}}=6$, $2=\sqrt{\frac{3}{x}}$, $x^2+.5x-7.5=0$, i.e., quadratic equations with decimal fractions as coefficients.
   d) Lack of understanding of basic concepts, e.g., students cannot explain why $1{,}273=1.273\times 10^3$.

It is not necessary to extend the foregoing list of illustrations to make it clear that even the simplest algebraic concepts and processes which doubtlessly receive a great deal of attention in mathematics seem to be forgotten by the time the student enters college. Similar evidence is offered in several studies in which pupils were tested a short time after they had finished secondary-school algebra.

As a solution of the problem of insufficient mathematical preparation for college sciences, three procedures have been followed by the college instructors.

1. Several studies have been made to identify the mathematical abilities and processes needed in the pursuit of physics, chemistry, and other sciences. The findings are used to construct tests which are administered to all students who sign up for the science courses. Those who fail are barred from admission until they give evidence that the mathematical deficiencies have been removed.

2. Some instructors, claiming that they cannot presuppose a knowledge of the mathematics needed in their courses, assume the responsibility of teaching facts and processes as need for them arises.

3. Whenever it is possible the use of mathematics is avoided, even if it involves the elimination of some aspects of the science courses which cannot be taught without mathematics.

The criticism that high-school graduates are poorly grounded in mathematics is worthy of serious consideration. It cannot be waved aside by saying that in all subjects students forget much of what has been taught and that there is no reason why mathematics should be an exception. The fact is that the various subjects are not comparable. Many facts in mathematics are taught for disciplinary and cultural purposes. Detailed information about them may be forgotten without reflection on the mathematics department. However, certain facts and processes are taught primarily because they are needed as tools in other subjects and in more advanced courses in mathematics. If they are not mastered to the point where they become permanent property of the pupil, instruction has failed.

Until the causes of failure are known, improvement cannot be expected. When a college student fails to add correctly $(a/n)+(b/n)$, the explanation is not that he has forgotten but that he has never acquired the correct understanding. The chances are that his performances at the time fractions were studied, even if the answers were right, were not based on insight and understanding but on imitation and memory. Hence the law of forgetting began to operate as soon as the topic was finished.

Nor should the blame for deficiencies in secondary-school mathematics be placed on the students. College students are a selected group as to mentality. They have passed successfully the first course in high-school algebra, a year's work in demonstrative ge-

ometry, and then a second course in algebra. At the end of each course the pupils at the lower end of the scale were eliminated. Further selection took place when they were admitted to college. Finally, the college students who elect courses in science and mathematics usually have an interest in mathematics and have shown ability in that field. These courses are generally avoided by those who feel that their mathematical preparation is inadequate. Evidently students who have come through the selective process cannot be classed as low in mathematical ability and the cause for mathematical deficiencies must be sought elsewhere.

Teachers of mathematics have recognized this and have been in search for ways of correcting the situation. Any device which may be expected to improve results is eagerly tried. This has created a good market for drill exercises, instructional tests, workbooks, and other devices which promise to help the student to master the subject. Unfortunately the results have not shown marked permanent improvement.

There is hope that a solution of the problem may grow out of certain recent changes of the secondary-school curriculum. The junior high school movement has made it possible to extend the period of secondary education downward into the seventh and eighth grades. An increasing number of elementary schools are modifying the curriculum for these grades to make it similar to the junior high school program. Opportunity is thus offered for a readjustment of the instructional materials traditionally presented in first-year algebra and demonstrative geometry. Some of it may be assigned to the eighth grade and some to the seventh. Not only will this offer relief to the overcrowded ninth- and tenth-grade courses, but the distribution of the subject matter over a longer period of time has the added advantage that it will slow up the learner's rate of progress and increase his chances for assimilation and genuine understanding. Furthermore, the close relations of arithmetic, algebra, and geometry provide broad mathematical experiences which can be made very helpful to the learner. Abstract principles and processes of algebra may be presented in the more concrete geometric settings. A solid foundation may be laid on which to build strong courses in algebra and geometry in the upper grades of the secondary school.

The downward movement of secondary-school mathematics should have the enthusiastic support of all teachers. The plan is only in the first stages of development. Much study and research will be required until the content of the new courses is as definitely and clearly defined as is the content of traditional algebra and geometry. When this has been accomplished the teachers of the upper grades will know exactly what they may presuppose and how they may continue without loss of time the work started in the lower grades. The goal will not be reached until the new program is widely introduced in seventh and eighth grades, irrespective of whether they are in an elementary school or in a junior high school.

Furthermore, the problem of method deserves the same careful study as that of selection and organization of instructional materials.

Students do not know how to use the mathematics taught in the secondary school.—The complaint is voiced by persons engaged in business and industry, and by teachers of college mathematics and the college sciences. It is also made by high-school teachers of subjects other than mathematics. Representatives of various high-school and college subjects were recently asked by the writer to list specific instances of inability of students in the use of mathematics. While some replied that they had no specific complaints to make, others expressed considerable dissatisfaction, and a list of difficulties was obtained from them. The following specific examples were selected as typical of the unfavorable comments.

1. The shop teacher told some high-school boys to divide an 8-foot board into 5 equal parts. They did not know how to figure it out.
2. The gymnasium teacher asked several boys to distribute a score of 5 among 3 schools which had tied in an athletic contest. They could not do it.
3. A teacher of chemistry finds that his students cannot see why the square of a given number less than 1 should be less than the given number.
4. The meaning of exponents is not clear. Therefore students cannot work the problem $\frac{432\times10^{-6}}{4\times10^{6}}$ correctly.
5. Students know how to solve $x=3y+6$ for y but are helpless in solving $s=at+b$ for t.
6. They have great difficulty in seeing that the formula $V=v_0+v_0\frac{t}{273}$ may be transformed into $V=v_0\left(\frac{273+t}{273}\right)$.

7. Students have the habit of avoiding thought. Thus, in the problem "If 65 g. of zinc give 2 g. of hydrogen, what will 10 g. of zinc give?" they will set up a proportion and go through the tedious process of solution in a formal manner when they could do it by simple, non-technical logic.
8. They cannot decide when a quantity is negligible. For example, they do not see that if x is negligible as compared with .1, the use of the quadratic in $\frac{(.1+x)x}{.1-x}=10^{-5}$ may be avoided.
9. They are unable to find the values of expressions like $31^{4.7}$.

The foregoing examples throw some light on the reasons why pupils do not know how to use the mathematics taught in the secondary schools. It is common experience to find that no matter how thoroughly a subject may be taught the strangeness of a new situation tends to confuse, and mistakes will result. It must be evident that when a high-school pupil is unable to divide an 8-foot board into 5 equal parts or to distribute a score of 5 over 3 schools it is not the mathematics but the situation which he is unable to understand and which causes the trouble. Practice in dividing 8 by 5 and 5 by 3 will not solve the problem. Moreover, for years he has had an abundance of practice with verbal problems. As early as in the third grade pupils are given similar exercises, the difference being in the situation and not in the manipulation. Instead of an 8-foot board it may be a chocolate bar that is to be divided equally. The best that mathematics teachers can do to improve problem-solving ability is to vary the situations and to see to it that they are understood.

The pupils in a solid-geometry class in a high school had been solving problems for months without any apparent difficulty. The problem situations were those of the farm, the shop, and the store. All problems were book problems with the data given and the required results were found by formulas and computation. One day while going through the pattern shop of the school the teacher saw a pile of patterns made by pupils. They were cylinders, prisms, pyramids, cones, spheres, and combinations, the very same solids that were studied in the course. It occurred to him that they would be excellent problem material for the solid-geometry class. He took them to the classroom and requested the pupils to find the surface area and the volume of each.

As simple as the problems were they caused considerable trouble. The reason was that no data were given and that they had to be found by the pupils. Thus to find the area of a triangular prism they had to construct the right section, measure the sides, and compute the perimeter before they could use the formula. Progress was temporarily blocked until this new element in the solution of the problems was cleared up. From then on no further difficulty was encountered. Pupils soon found their own ways of measuring the altitudes of pyramids and the diameters of spheres. It seems, therefore, that two important steps should be provided for by teachers of mathematics in training pupils to solve problems: There must be problems dealing with a variety of situations, and new situations which are confusing to the pupils must be cleared up by the teacher. Likewise, unfamiliar situations which occur in problems met in subjects other than mathematics must be explained by the teachers of those subjects.

In a social-science class the pupils were to solve the following problem: "In an election 16,248 votes were cast for one candidate and 12,314 for his opponent. How many votes were cast?" When the answers came in some pupils had subtracted. They were asked to explain their method of solution and said, "We thought cast means casting out and that means subtracting." Thus, the mistake was due to the fact that the situation had not been made clear to them. Perhaps it was a vocabulary difficulty.

An analysis of the problem materials of three arithmetic books disclosed 450 different ways of expressing addition and almost as many ways of indicating subtraction. It is impossible to teach all of them in arithmetic. Hence, teachers of subjects in which problems calling for addition or subtraction arise must keep in mind that the newness of a situation may be the cause of incorrect solutions unless they make the situation clear to the pupils.

It must be stated emphatically that from the fact that problems of types 1, 2, 5, and 6 are difficult it does not follow that the pupils were poorly taught in mathematics. The real reason might be lack of familiarity with the problem situations and the terminology. Sometimes the symbols contained in the formula are new. If the pupil knows how to solve $x=3y+6$ it should be no trouble for the teacher of physics to show that the solution of $s=at+b$ for t is

really the same and that it involves no new processes. The fact is that when the meaning of the symbols s and t are not understood the student may think of them as abbreviations of the words "space" and "time" and that the equation $s=at+b$ does not look to him like an algebraic equation, as $x=3y+6$. The trouble with transforming $v_0+v_0(t/273)$ to $v_0[(273+t)/273]$ may not be caused by lack of understanding of the laws of addition and factoring but by failure to understand the meaning of the symbols. The teacher of physics who thoughtlessly disposes of the problem by saying that the student does not know how to use his mathematics may in fact be guilty of teaching physics badly. He is expected to discover the real difficulty and render some assistance.

Thus the attainment of best results in the use of mathematics in another subject is a co-operative enterprise of the teacher of mathematics and of the teacher of the second subject. This assertion does not deny that much poor teaching of mathematics exists. Indeed, it cannot be overemphasized that the best defense of the subject is to improve the teaching. The problem material should be enriched, the situations should be varied, and the symbols should be made clear. Nothing should be left undone to train pupils to attack problems containing new situations with confidence and success. Variety of experiences is most essential.

It is not necessary to go outside of the field of mathematics to become aware of the importance of variety. If a college student makes a mistake in solving $2x=14$, the probability is that he never had a clear conception of the relationship involved in the equation. His high-school teachers of mathematics will have to take the blame for that. They may have drilled him by making him solve long lists of equations like $3x=15$, $7x=21$, $12x=96$, etc., never realizing that ability to give the correct results is not a reliable indication of understanding. Most likely after solving a few problems the pupil forgot all about the equation and merely divided 15 by 3, 21 by 7, and 96 by 12. The more drill he was given the farther removed were his thoughts from the equation. What he really needed was less drill and more stress on the uses of the equation. He should be given percentage problems leading to $.04p=14$, circumference problems leading to $2\pi r=14$, lever problems leading to $5x=2\times 7$, and many other types of problems. In

each new situation he should give thought to the mathematical relationship and the process to be employed in the solution of the equation. Purely mechanical imitation and performance should be out of question. The outcome will be complete understanding and permanent retention.

When pupils have had the right kind of teaching it does not follow that new situations will cause no confusion, but that in general the difficulty will be less serious. Thus students of trigonometry may fail to recognize that $2 \sin^2 x - 3 \sin x + 1 = 0$ is a quadratic equation. Hence they may not be able to solve it, although they may solve $2x^2 - 3x + 1 = 0$ without difficulty. The implication, therefore, is not that they have been poorly taught but rather that they need further experiences to attain complete understanding of a quadratic equation. It is the duty of the teacher of trigonometry to contribute his share to bring about this understanding. Each new situation offers an opportunity for a review and a new view.

It has been shown that inability to use secondary-school mathematics when needed in other subjects in the high school and college and in the more advanced mathematical courses places certain responsibilities upon the teachers of these subjects and courses, which cannot be evaded by blaming the teacher of high-school mathematics. Furthermore, some of the problems in the foregoing list deal with a type of mathematics which receives practically no attention in the lower courses. Thus the idea of negligible quantity in problem 8 would have to be derived from measurement for which very few schools have the apparatus and other necessary facilities. If developed there would be little or no use for it in the further study of mathematics. It should therefore be developed in the science laboratory when actual need for it arises. Verbal problems like that in exercise 7 are generally worked by proportions, i.e., by algebraic methods rather than by the arithmetical method. Whenever teachers of science find that the arithmetical method is simple and saves work and time, they should take the trouble of explaining the method thoroughly to the class. The problem in exercise 9 which calls for the value of $31^{4.7}$ is generally discussed in mathematics but there is practically no further use for it. Lack of practice will therefore reduce retention. The science teachers will have to explain and reteach some problems when they occur.

Responsibility for problems like the fourth, to find the value of $(432\times10^{-6})/(4\times10^{6})$, rests entirely on the teacher of mathematics. The processes involved in the solution belong to the study of algebra and should be mastered by the pupils.

The answers to the criticism that students are not able to use the mathematics taught in the secondary school may now be summarized briefly as follows:

1. The mathematical facts used in science which are not taught in mathematics and those which are not sufficiently used in mathematics to help the student attain mastery should be taught or carefully retaught by the science teacher.

2. Non-mathematical terms and situations which occur in other school subjects and which are new to the student must be carefully explained to the point where they are thoroughly understood. Unfamiliarity is often the cause of mathematical difficulty and errors.

3. Teachers of mathematics must provide the widest possible applications for the mathematical facts and principles which are to be mastered. This may easily be done by correlating the mathematical subjects with each other and with the other school subjects. The tendency of teaching facts by themselves to the exclusion of all other facts is harmful to mathematics.

The place of mathematics in the changing curriculum.—So far attention has been given to criticisms directed against mathematics. It is not to be inferred that dissatisfaction with education is limited to the mathematical subjects. There is a growing demand for far-reaching changes in the entire school curriculum. The schools are being criticized because they have failed to give youth and adults an understanding of modern life and social conditions. The social-science studies in particular are blamed for having fallen short of educational objectives. Some critics go so far as to say that the money spent on high-school education is practically thrown away.

Tendencies are now developing which are expected to secure better results. One is to allot additional time to the social sciences —in fact, to make them the most important part of the curriculum. The other is to make corresponding reductions of the time allotted to other subjects. It is hoped, however, that the time may be regained by elimination of duplication and overlapping in all subjects, the social sciences included. Part of the plan is to break down the

artificial barriers which separate the various high-school subjects and to integrate subject matter which really belongs together, but which has become separated and is being presented in different courses. It is expected that this integration will be beneficial to all subjects, that it not only will make up for any time reductions but also bring about a gain in effectiveness of teaching, and that it will create a type of education adapted to modern life and civilization.

For example, the training in oral and written language may be made an important function of every teacher and every department if it is closely tied up with all school subjects and with the extra-curricular activities of the pupils. The English department will no longer assume the entire responsibility for such training. It will supplement it and provide organized systematic instruction in language whenever the needs become too great to be taken care of by the other departments. It will therefore establish close co-operation with all of them.

In the social sciences the plan of integration is to unite into a general social-science course a large body of instructional materials now presented in such separate courses as history, geography, economic society, and modern problems. In the natural sciences much progress along the lines of integration has been made during recent years. General science courses are now thoroughly established.

Experimentation by high schools with new and reconstructed curriculums is being undertaken on a large scale. It is being done with the consent and even the encouragement of leading colleges and universities, which removes one of the greatest obstacles to curriculum reconstruction. Over two hundred colleges and universities have recently agreed to admit pupils coming from the experimental schools without requiring the regular entrance examinations.

It is an interesting historical fact that teachers of mathematics were the first to study the problem of breaking down artificial divisions which separate the various mathematical subjects. The recommendations of the Committee of Ten, of Perry, Klein, and Moore, which have been widely and enthusiastically accepted, were all advocating some type of correlation. The movement was also indorsed by the National Committee on Mathematical Requirements. Thus to the teachers of mathematics the development of

this tendency in other school subjects should be of particular interest. In mathematics it has led to the construction of a variety of courses designated as correlated, general, fused, unified, composite, co-operative, and combined mathematics. The differences in the names need not disturb one. They indicate different ways of attacking the same problem. The common purpose is to gain for the subject the advantages to be derived from doing away with artificial divisions.

The greatest progress has been made in the reorganization of the courses offered for Grades VII and VIII. Modern writers do not designate textbooks for these grades as arithmetics, algebras, and geometries. Such titles as *Seventh-Year Mathematics*, *Modern Mathematics*, Book I, *Junior Mathematics*, etc., are the rule rather than the exception. Writers of college textbooks also are giving much thought to the unification of subject matter in college mathematics. Evidence of this is a list of textbooks for junior colleges, such as the *Course in Mathematics*, by Bailey and Woods; *Introduction to Mathematical Analysis*, by Griffin; *Elementary Functions*, by Gale and Watkeys; *Unified Mathematics*, by Karpinsky, Benedict, and Calhoun; and the two-volume series by Professor Mayme I. Logsdon entitled *Elementary Mathematical Analysis*. In this most recent contribution Mrs. Logsdon presents the materials of trigonometry, college algebra, and analytical geometry arranged with regard to their relations to each other and not in the usual ways as three separate and distinct doctrines. The ideas of calculus are introduced early.

Thus the movement of correlation has gained considerable momentum in junior high school and in junior college mathematics. Moreover, even the courses designated as "algebra" and "geometry" contain commonly an abundance of instructional materials drawn from the other mathematical subjects.

It seems, therefore, that from the standpoint of education in general as well as from the standpoint of mathematics the need for the correlation of the various mathematical subjects was never more apparent than at the present time. Better mathematical training will be obtained from the study of mathematics than from the successive study of arithmetic first, then algebra, and then geometry.

High-school science and mathematics have much in common. The sciences furnish many helpful formulas which may be used to give training in mathematical calculation, graphical representation, and functional thinking. History shows that the union of mathematics and science has been an important factor in the development of both. Many valuable facts and principles of mathematics have been discovered by men investigating problems in science. Indeed, physics and chemistry are sometimes regarded as branches of mathematics. The tendency in education to integrate high-school subjects should stimulate the correlation of mathematics and science.

The unification of the mathematical subjects.—Correlation extends to the teacher and student of mathematics certain valuable advantages which are lost when each of the mathematical subjects is taught by itself to the exclusion of the others. When several subjects are touched upon, it is possible to arouse the interests of a larger number of pupils. When the simple facts of the various subjects are brought together at the beginning of a course, a psychological arrangement is facilitated. Instruction and subject matter may be adapted to the ability of the learner. Complexity may be increased gradually because the content of each subject is being distributed over a longer period of time. The pupil's rate of progress is not too rapid for assimilation. The abstract processes of algebra may be made concrete by use of geometric material and the use of algebraic symbols is helpful in the study of geometry. Since none of the subjects is dropped entirely for any great length of time, the number of necessary repetitions and reviews is being greatly, reduced and time is saved. Familiarity with the materials of several subjects increases the pupil's resourcefulness and mathematical power.

In view of the foregoing advantages it seems that the movement of correlation has not developed as rapidly as should be expected. The reason is that certain factors operate heavily against its progress. Teachers are naturally conservative in introducing innovations. Some are afraid that the combination of several subjects might increase rather than diminish the complexity and that it will introduce additional difficulties in teaching and learning. Others admit that there are many points of contact but insist that there

are phases which make a correlation unnatural and detrimental. Still others fear that the subjects of algebra and geometry will lose their individualities, which might endanger the cultural values of mathematics. Finally, many teachers believe that thoroughness is best attained if each subject is studied intensively for a long period of time to the exclusion of the others.

Thus the unification of the mathematical subjects is being questioned. However, the measured results that are available seem to favor the plan. In general, the investigations that have been made establish the fact that the results obtained with combined courses are at least as good as those obtained with the separate courses when measured with tests in algebra and geometry. However, when mathematical power tests are used, as, for example, tests of ability to use and apply mathematics and tests of functional thinking, the results obtained with the combined courses indicate superiority.

While the cause of correlation has been greatly promoted by some school administrators, it has suffered from the opposition of others. Teachers anxious to improve their work have complained that they are placed on the defensive as soon as they try out the new plan. Others report that their principals who formerly criticized the department severely because the results did not satisfy them were loud in objecting to changes, even when they gave promise of better results.

Changes in the curriculum are likely to bring to the surface certain administrative problems. The question of transfer is usually one of the first to be brought up. However, it should be kept in mind that there is enough difference in textbooks in algebra and geometry to raise the same question but not much thought is given to it because the names of the courses are the same. In justice to the pupil every transfer calls for consideration and adjustment. Sometimes it is argued that the teachers are not qualified to teach the new courses. There seems to be a feeling that a teacher may be able to teach algebra and geometry but not be able to teach mathematics, and the fact is overlooked that the correlation does not introduce any new mathematical materials. Some administrators object to correlation because it might disturb the peace of the teaching staff, and still others claim that they are against any change because some of the teachers refuse to co-operate in any new

scheme. None of these arguments needs to be taken seriously. In reality they are criticisms of the administration rather than the teaching staff. Thus, it is very doubtful whether the administrator who is willing to excuse the teacher who cannot be induced to improve his courses will excuse disorder in a classroom because he cannot convince the teacher that good order is essential.

Summary and recommendations.—Secondary education is being criticized because it is said to have failed to give an understanding of modern social conditions. Attempts are being made to change the curriculum to accomplish more satisfactory results. Mathematics, like the other subjects, has received its share of criticisms. In particular they are: that too large a percentage of pupils fail in the subject, that they are unable to retain what has been taught, and that they cannot use mathematics when need for it arises in other subjects.

A search of the literature as to recommendations for improving mathematics discloses that the best suggestions from which results may be expected are found in the recommendations of the Committee of Ten, of Perry of England, of Klein of Germany, and of Moore of Chicago. They advised: that systematic instruction be given in concrete and experimental geometry and in algebraic symbols and simple equations during the period preceding the senior high school; that algebra and geometry of this period not be taught as separate subjects but in connection with arithmetic; that the useful parts of the various mathematical subjects be introduced early; and that emphasis be placed on the applications of mathematics in other school subjects.

An outstanding feature of these recommendations is that they lean toward the correlation of the mathematical subjects with each other and with the natural sciences. Attention was called to the advantages of, and the objections to, the unification of subjects. Measured results are favorable to it.

Mathematics does not stand alone in the attempt to improve conditions by the method of unification. Courses in general science seem to be well established. A similar tendency is developing among the social studies.

In this period of educational unrest and experimentation teachers of mathematics cannot afford to be indifferent. Criticisms should

no longer be waved aside as of little or no importance. When they are found to be justifiable, everything possible should be done to remove cause for further criticism. When the fault lies with the subjects in which mathematics is used, teachers should enlist the co-operation of the other departments.

An excellent beginning in the reconstruction of mathematics has been made in the newer junior high school courses. The movement deserves the whole-hearted support of all teachers of mathematics. It should be extended to include the work of seventh and eighth grades in general, and the content of the courses should be standardized. It should form a gradual introduction to the subject of mathematics; establish clear meanings of the basic concepts; and comprise the mathematics which people in general need to know. Geometry should be the unifying element in this introductory work. Algebra should be developed as it is found helpful and convenient.

On this foundation the reconstructed upper courses should be built. Ample training in arithmetical computation should be provided in all courses. Emphasis should be at first on algebra and then on geometry. There will be sufficient time to offer the practice necessary to attain mastery of the important processes and facts.

Attention should be given to the mathematical needs in the other school subjects and every attempt should be made to establish the sympathetic co-operation of the teachers.

From this reconstruction much is to be gained for the courses in demonstrative plane and solid geometry. The attacks on the traditional courses are gaining in severity. It is said that the informational parts of geometry may be more quickly found by the intuitive method than by logical reasoning and that the particular type of reasoning which is developed with so much effort is of little value to people in everyday activities. Indeed, it is said that it is not even used in the other mathematical courses. Under these conditions, it is argued, the course in tenth-grade geometry cannot be justified much longer, especially with the growing demand for more emphasis on the social studies. Hence pupils are being advised not to elect it if they can avoid it.

A plan which moves demonstrative geometry upward to the Junior or Senior years has much in its favor. At that level it is

possible to develop an appreciation of the beauty of logical reasoning which builds up a complete system of theorems from a few facts taken for granted. Such a course can be made very valuable and interesting. It will appeal to students as one of the most interesting school studies. Its major aim, however, is not the practical, i.e., to acquire geometric information, but the cultural and disciplinary. In a school offering such a course it would be open to Juniors and Seniors only.

Trigonometry will be in a position similar to that of geometry. Some trigonometric ideas will be developed in the early courses, but the subject of trigonometry will be reserved for those Juniors and Seniors who have developed a genuine interest in mathematics and have shown sufficient mathematical ability to profit from the study of the subject.

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